MATH 252 --- Calculus III
Spring 2025

Instructor: Bo-Wen Shen, Ph.D.

Lecture #23a:
Curl and Divergence in the Two Dimensional Space

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Learning Outcomes

Formulas for Grad, Div, Curl, and the Laplacian

	Cartesian (x, y, z) **i**, **j**, and **k** are unit vectors in the directions of increasing x, y, and z. **P, Q, and R** are the scalar components of $\mathbf{F}(x, y, z)$ in these directions.
Gradient	$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$
Divergence	$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$
Curl	$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$
Laplacian	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

The Fundamental Theorem of Line Integrals

1. Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ be a vector field whose components are continuous throughout an open connected region D in space. Then there exists a differentiable function f such that

$$\mathbf{F} = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

if and only if for all points A and B in D the value of $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path joining A to B in D.

2. If the integral is independent of the path from A to B, its value is

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Green's Theorem (Tangential Form)

$$\iint_R \nabla \times \vec{F} \cdot \vec{k}\, dxdy = \oint \vec{F} \cdot d\vec{r}$$

Green's Theorem (Normal Form)

$$\iint_R \nabla \cdot \vec{F}\, dxdy = \oint \vec{F} \cdot \vec{n}\; ds$$

A "Meta" Vector: $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ TBD

- Consider a "meta" vector $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, a function $f = f(x, y, z)$ and a vector $F = (P(x, y, z), Q(x, y, z), R(x, y, z))$.

We can define the following: $\nabla: nabla$

- Gradient:

$$\nabla f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (f_x, f_y, f_z)$$

- Curl (a Cross product of ∇ and $\vec{F}$):

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z, P_z - R_x, Q_x - P_y)$$

- Divergence (a Dot product of ∇ and $\vec{F}$):

$$\nabla \bullet F = (P_x + Q_y + R_z)$$

Curl: 2D Version, $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

- Curl (a Cross product of ∇ and $\vec{F}$, $\vec{F} = (P(x,y), Q(x,y), 0)$.

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix}$$

$$= i\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q(x,y) & 0 \end{vmatrix} - j\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P(x,y) & 0 \end{vmatrix} + k\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P(x,y) & Q(x,y) \end{vmatrix}$$

$$= i\,\frac{-\partial Q(x,y)}{\partial z} + j\,\frac{\partial P(x,y)}{\partial z} + k\left(\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y}\right)$$

$$= k\left(Q_x - P_y\right) = \left(0, 0, Q_x - P_y\right)$$

one non-zero component

Curl: 2D Version

Curl (a Cross product of ∇ and $\vec{F}$)

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix}$$

$$= k\left(Q_x - P_y\right)$$

Divergence: 2D Version

Divergence (a Dot product of ∇ and $\vec{F}$)

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot (P, Q)$$

$$= \left(\frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y}\right)$$

$$= P_x + Q_y$$

$$\begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ \Updownarrow & \Updownarrow \\ P(x,y) & Q(x,y) \end{vmatrix}$$

Divergence and Curl: 2D Version

Divergence (a **D**ot product of ∇ and $\vec{F}$)

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot (P, Q)$$

$$= \left(\frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y}\right)$$

$$= P_x + Q_y$$

$$\nabla \cdot \vec{F} = \quad \frac{\partial}{\partial x} \qquad \frac{\partial}{\partial y}$$

$$\qquad\quad \updownarrow \qquad\qquad \updownarrow$$

$$\qquad\quad P(x,y) \qquad Q(x,y)$$

Curl (a **C**ross product of ∇ and $\vec{F}$)

$$\nabla \times \vec{F} = k\left(Q_x - P_y\right)$$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix}$$

Webassign

We define a meta vector as follows: $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$,

and have $f(x,y)$ and $\vec{F} = (P(x,y), Q(x,y))$.

Type equation here.

(a) Find the gradient of $f(x,y)$, $\nabla f = \left(f_x \;,\; f_y \right)$;

(b) Compute the curl of $\vec{F}$ *as a cross product of* ∇ *and* $\vec{F}$, $\nabla \times \vec{F} = \left(Q_x - P_y \right) \vec{k}$;

(c) Compute the divergence of $\vec{F}$ *as a dot product of* ∇ *and* $\vec{F}$, $\nabla \cdot \vec{F} = P_x + Q_y$;

Curl: 2D Version

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix}$$

$$= k\left(Q_x - P_y\right)$$

$$\vec{F} = (2y, -2x) = (P, Q)$$

$$P = 2y \qquad P_y = 2$$

$$Q = -2x \qquad Q_x = -2$$

$$\nabla \times \vec{F} = k\left(Q_x - P_y\right) = (-4)k$$

A "Meta" Vector: $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$

- Consider a "meta" vector $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$, a function $f = f(x, y)$ and a vector $\vec{F} = (P(x,y), Q(x,y))$

We can define the following:

$\nabla: nabla$

- Gradient:

$$\nabla f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(f_x, f_y\right)$$

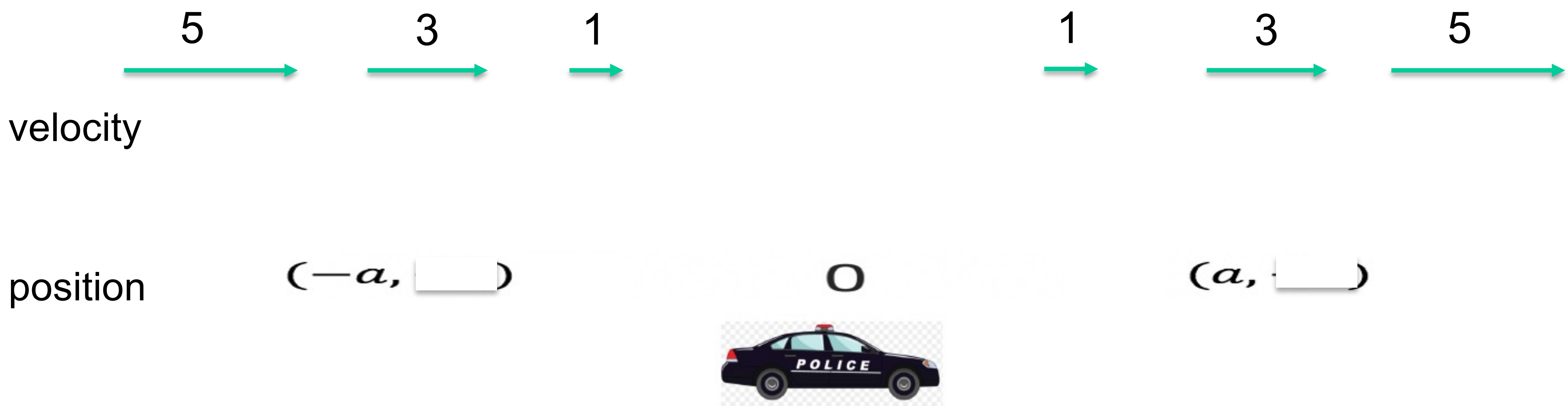
- Curl (a Cross product of ∇ and $\vec{F}$):

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = k\left(Q_x - P_y\right)$$

- Divergence (a Dot product of ∇ and $\vec{F}$):

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot (P, Q) = P_x + Q_y$$

Flows: Divergence vs. Convergence



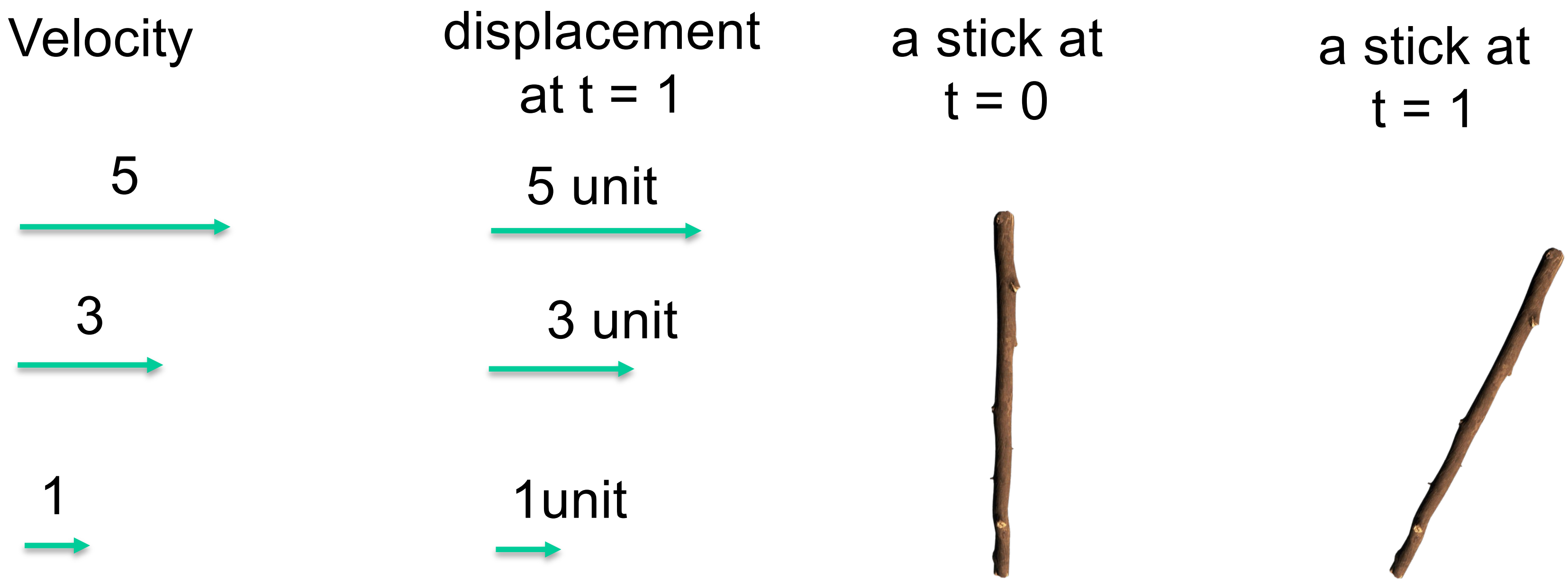
Let velocity $\vec{F}$ be the vector field, $\vec{F} = (|x|, 0)$, i.e.,

- $\vec{F} = (x, 0)$ for $x \geq 0$ and
- $\vec{F} = (-x, 0)$ for $x < 0$.

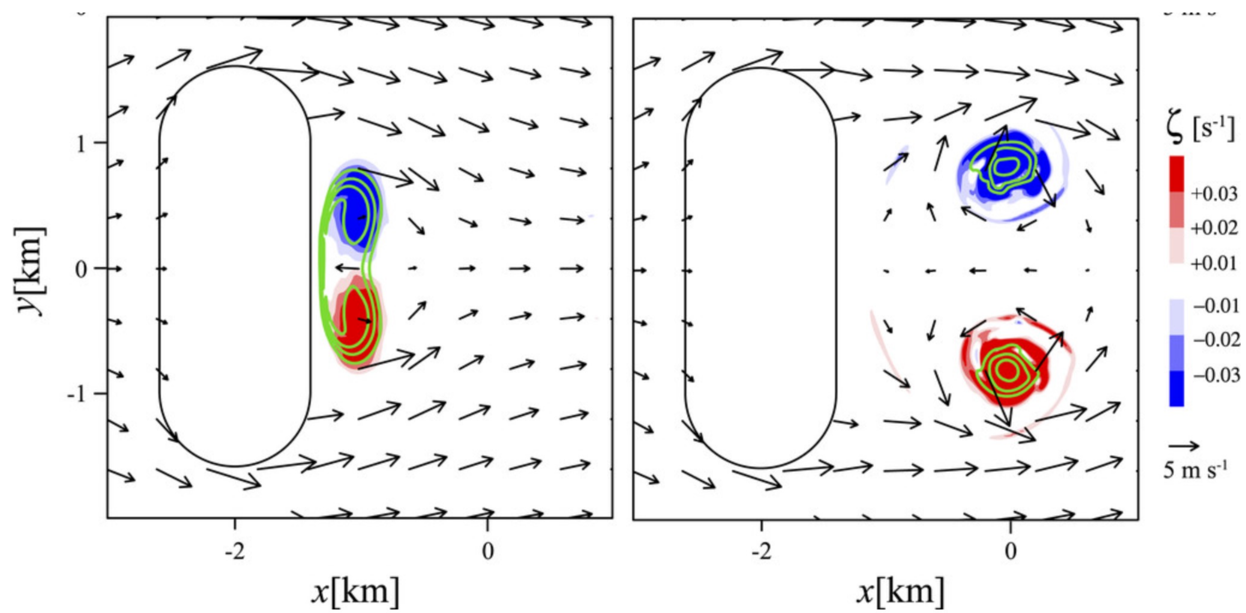
$\nabla \cdot \vec{F} = P_x + Q_y = 1 > 0$, divergence for $x > 0$

$\nabla \cdot \vec{F} = P_x + Q_y = -1 < 0$, convergence for $x < 0$ traffic jam

Flows: Curl and "Shear"



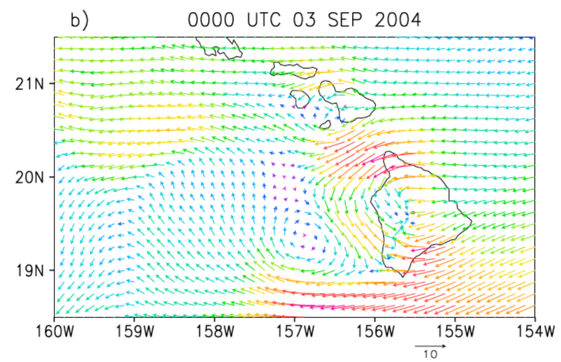
- $\vec{F} = (\mathrm{U}, \mathrm{V}) = (y, 0)$
- Curl: $V_x - U_y = 0 - 1 < 0,$ negative (clockwise)



negative

positive

(Rotunno et al.)



Hawaiian Vorticities
(Shen et al. 2006)

Figure 3. Simulations of the Hawaiian Vortices initialized at 0000 UTC 1 September 2004. (a) 36 h simulation and (b) 48 h simulation.