MATH 252 --- Calculus III
Spring 2025

Instructor: Bo-Wen Shen, Ph.D.

Lecture #25:
Sections 16.3
The Fundamental Theorem for Line Integrals

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Supplemental Materials

The supplemental materials with a summary in Table 1 are provided to help students review the following topics:

(1) vector fields; (2) gradient and normal vector; (3) curl and circulation;

(4) divergence and flux; (5) line integrals; (6) double integrals;

(7) fundamental theorem of line integrals;

(8) conservative fields and independence of path;

(9) Green's theorem in both the tangential and normal forms;

(10) a comparison amongst Green's, Stokes' and Divergence theorems.

Learning Outcomes

Formulas for Grad, Div, Curl, and the Laplacian

	Cartesian (x, y, z) **i**, **j**, and **k** are unit vectors in the directions of increasing x, y, and z. **P, Q, and R** are the scalar components of $\mathbf{F}(x, y, z)$ in these directions.
Gradient	$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$
Divergence	$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$
Curl	$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$
Laplacian	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

The Fundamental Theorem of Line Integrals

1. Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ be a vector field whose components are continuous throughout an open connected region D in space. Then there exists a differentiable function f such that

$$\mathbf{F} = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

if and only if for all points A and B in D the value of $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path joining A to B in D.

2. If the integral is independent of the path from A to B, its value is

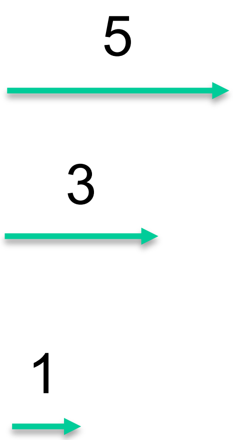
$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Curl & Divergence (2D Version)

Curl:
"a **C**ross product of ∇ and F"

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix}$$

$$\nabla \times \vec{F} = k\left(Q_x - P_y\right)$$



Divergence:
"a **D**ot product of ∇ and F"

$$\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z}$$

$$P(x,y) \quad Q(x,y) \quad 0$$

$$\nabla \cdot F = P_x + Q_y$$

16.3 The Fundamental Theorem for Line Integrals

We will discuss the following topics:

- anti-derivative vs. potential function;
- how to find the f of ∇f ➔ Find a potential function f;
- Independence of path (for line integrals);
- conservative property (for vector fields).

For Mid Term II

One Formula Summary for Sections 14.4 – 14.8

(14.4) $\mathrm{d}f = f_x dx + f_y dy = \nabla f \cdot \mathrm{d}\vec{r}$ total differential

$$calc\ I:\ \frac{df(x)}{dx} = f'(x) = f_x$$

$$df = f'(x)dx = f_x dx$$

$$calc\ III:\ df(x,y) = f_x dx + f_y dy$$

16.2 Line Integrals→ Section 16.3

$\vec{F} = (P, Q, R)$

$\vec{T}$ tangent vector $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$

$$\mathbf{W} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$ The definition

$$= \int_C \mathbf{F} \cdot d\mathbf{r}$$ Vector differential form

$$= \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt$$ Parametric vector evaluation

$$= \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt} \right) dt$$ Parametric scalar evaluation

$$= \int_a^b P dx + Q dy + R dz$$ Scalar differential evaluation

$$\stackrel{?}{=} \int_a^b \nabla f \cdot d\vec{r} = \int_a^b df = f(b) - f(a)$$ Section 16.3 ("anti-derivative")

16.3 Anti-derivative vs. Potential Function

1 Variable $\dfrac{dG}{dx} = G'(x) \quad \Rightarrow dG = G_x(x)dx$

$\Rightarrow G_x dx = dG \qquad G$: “anti-derivative”

$$\Rightarrow \int_a^b G_x(x)dx = \int_a^b dG = G(b) - G(a)$$

2 Variables $\mathrm{d}f = f_x dx + f_y dy = \nabla f \cdot \mathrm{d}\vec{r}$

$\Rightarrow \nabla f \cdot \mathrm{d}\vec{r} = df \qquad f$: “potential function”

$$\Rightarrow \int_a^b \nabla f \cdot \mathrm{d}\vec{r} = \int_a^b df = f(b) - f(a)$$

16.3 The Fundamental Theorem for Line Integrals

- The Fundamental Theorem of Calculus

1 Variable $$\int_a^b G'(x)dx = G(b) - G(a)$$ The Net Change Theorem

  - The integral of a rate of change (G') is the net change.

- The Fundamental Theorem for Line Integrals:

2 Theorem Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

2 or 3 Variables $$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

  - The line integral of ∇f is the net change.

$$df = f_x dx + f_y dy$$ The total differential of a function f

Example: Find a Potential Function

1 Variable $\frac{dG}{dx} = G'(x)dx \quad \Rightarrow dG = G_x(x)dx$

$G_x = 2x \qquad G = x^2$: "anti-derivative"

$$\Rightarrow \int_a^b 2x\, dx = G(b) - G(a) = b^2 - a^2$$

2 Variables $\mathrm{d}f = f_x dx + f_y dy \; = \nabla f \cdot \mathrm{d}\vec{r}$

$\nabla f = (2x, 2y) \qquad f$?:

$$\Rightarrow \int_a^b \nabla f \cdot \mathrm{d}\vec{r} = \int_a^b df \; = f(b) - f(a)$$

Integration "Constants"

$$f = f(x) \qquad g = g(x, y)$$

$$f_x = 2x \qquad g_x = 2x$$

$$f(x,) = x^2 + C \qquad g(x, y) = x^2 + K(y)$$

Verify

Verify

$$f_x = 2x \qquad g(x, y)_x = 2x$$

Keep y as a constant

Example: Find a Potential Function

$\nabla f = (2x, 2y)$ $\quad$ f?:

$f_x = 2x$ $\quad$ $f_y = 2y$

$f = x^2 + g(y)$ $\quad$ ↓

$f_y = g_y(y)$ → $g_y(y) = 2y$

$f = x^2 + g(y)$ ← $g(y) = y^2 + C$

$f = x^2 + y^2 + C$

Example: Anti-derivative vs. Potential Function

2 Variables $\mathrm{d}f = f_x dx + f_y dy = \nabla f \cdot \mathrm{d}\vec{r}$

$\nabla f = (2x, 2y)$ $\quad f = x^2 + y^2 + C$

point b: (k, l)

point a: (i, j)

$$\Rightarrow \int_a^b \nabla f \cdot \mathrm{d}\vec{r} = \int_a^b df = f(b) - f(a) = k^2 + l^2 - i^2 - j^2$$

Summary: Anti-derivative vs. Potential Function

1 Variable $\frac{dG}{dx} = G'(x)dx \quad \Rightarrow dG = G_x(x)dx$

$G_x = 2x \qquad G = x^2$: "anti-derivative"

$$\Rightarrow \int_a^b 2x\,dx = G(b) - G(a) = b^2 - a^2$$

2 Variables $\mathrm{d}f = f_x dx + f_y dy \;= \nabla f \cdot \mathrm{d}\vec{r}$

$\nabla f = (2x, 2y) \qquad f = x^2 + y^2 + C$

point b: (k, l)

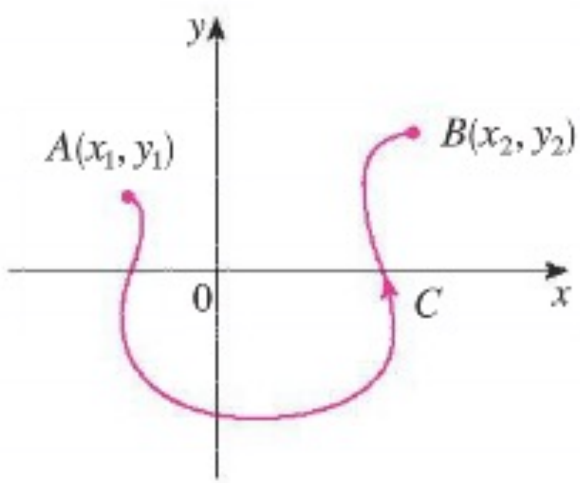
point a: (i, j)

$$\Rightarrow \int_a^b \nabla f \cdot \mathrm{d}\vec{r} = \int_a^b df \;= f(b) - f(a) \;= k^2 + l^2 - i^2 - j^2$$

16.3 2D vs. 3D

2 Theorem Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$



$$\int_c \nabla f \cdot dr = \int_c df = f(b) - f(a)$$

$$df = f_x dx + f_y dy$$

The total differential of a function f

2 var $\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2) - f(x_1, y_1)$

3 var $\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$

13 Definition Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Then the **line integral of F along** C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

PROOF OF THEOREM 2

$when\ F = \nabla f,$

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$\nabla f \cdot \mathrm{d}\vec{r}$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt$$

$$= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\, dt \qquad \text{(by the Chain Rule)}$$

$$= \mathrm{d}f$$

$$= f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Potential Function

EXAMPLE 5 If $\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + (2xy + e^{3z})\,\mathbf{j} + 3ye^{3z}\,\mathbf{k}$, find a function f such that $\nabla f = \mathbf{F}$.

$$\nabla f = \left(f_x, f_y, f_z\right) = (y^2, 2xy + e^{3z}, 3ye^{3z})$$

$$f_x = y^2 \qquad f_y = 2xy + e^{3z} \qquad f_z = 3ye^{3z}$$

$$f = xy^2 + g(y, z)$$

$$f_y = 2xy + g_y(y, z) \quad \Rightarrow \quad g_y(y, z) = e^{3z}$$

$$g = ye^{3z} + h(z)$$

$$f = xy^2 + ye^{3z} + h(z)$$

$$f_z = 3ye^{3z} + h_z(z) \quad \Rightarrow \quad h_z(z) = 0$$

$$h = K$$

$$f = xy^2 + ye^{3z} + K$$

16.3 The Fundamental Theorem for Line Integrals

EXAMPLE 1 Find the work done by the gravitational field

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$

in moving a particle with mass m from the point (3, 4, 12) to the point (2, 2, 0) along a piecewise-smooth curve C. (See Example 4 in Section 16.1.)

$$\mathbf{F}(x, y, z) = \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{k}$$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

whether $F = \nabla f$?

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = \int_a^b df = f\Big|_a^b$$

16.3 The Fundamental Theorem for Line Integrals

EXAMPLE 1 Find the work done by the gravitational field

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$

in moving a particle with mass m from the point (3, 4, 12) to the point (2, 2, 0) along a piecewise-smooth curve C. (See Example 4 in Section 16.1.)

$$\vec{F} = (P, Q, Z) = \nabla f$$

$$f = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = \int_a^b df = f\Big|_a^b$$

$$= f(2, 2, 0) - f(3, 4, 12) = \frac{mMG}{\sqrt{2^2 + 2^2}} - \frac{mMG}{\sqrt{3^2 + 4^2 + 12^2}} = mMG\left(\frac{1}{2\sqrt{2}} - \frac{1}{13}\right)$$

The work done by F along any curve joining the two points and not passing through the origin is the same. ➔ independence of path

Revisit : Anti-derivative vs. Potential Function

1 Variable $\frac{dG}{dx} = G'(x)dx \quad \Rightarrow dG = G_x(x)dx$

$G_x = 2x \qquad G = x^2$: "anti-derivative"

$$\Rightarrow \int_a^b 2x\,dx = G(b) - G(a) = b^2 - a^2$$

2 Variables $\mathrm{d}f = f_x dx + f_y dy = \nabla f \cdot \mathrm{d}\vec{r}$

$\nabla f = (2x, 2y) \qquad f = x^2 + y^2 + C$

point b: (k, l)

point a: (i, j)

$$\Rightarrow \int_a^b \nabla f \cdot \mathrm{d}\vec{r} = \int_a^b df = f(b) - f(a) = k^2 + l^2 - i^2 - j^2$$

TBD

if and only if

Independence of paths
conservative

$\overrightarrow{F} = \nabla f$ on D

if and only if

$\oint \vec{F} \cdot d\vec{r} = 0$

$\nabla \times \overrightarrow{F} = 0$ throughout D

conservative

2D: $\nabla \times \vec{F} = k\left(Q_x - P_y\right) = 0$

$\nabla \times \nabla f = 0$

16.5: A "Meta" Vector: $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

- Consider a "meta" vector $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, a function $f = f(x, y, z)$ and a vector $F = (P(x,y,z), Q(x,y,z), R(x,y,z))$.

We can define the following:

- Gradient:

$$\nabla f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (f_x, f_y, f_z)$$

- Curl (a Cross product of ∇ and $\vec{F}$):

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z, P_z - R_x, Q_x - P_y)$$

- Divergence (a Dot product of ∇ and $\vec{F}$):

$$\nabla \bullet F = (P_x + Q_y + R_z)$$

16.3 Independence of Path

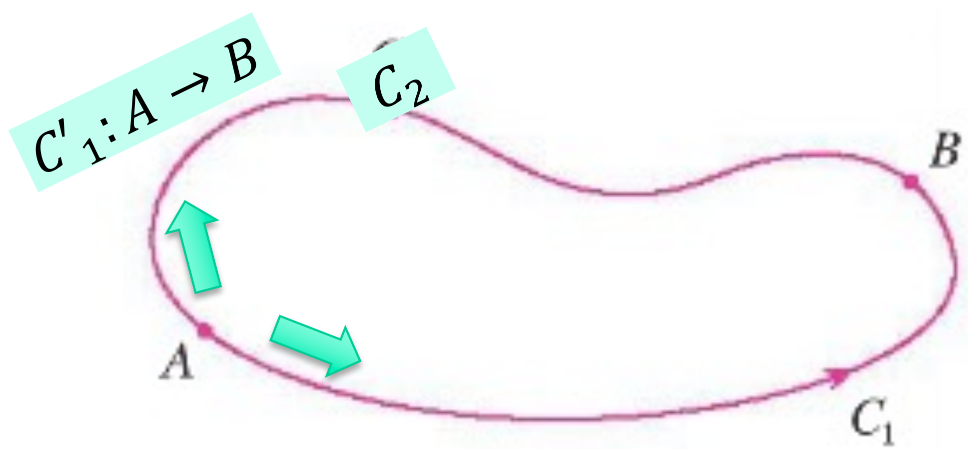
$\int_C \mathbf{F} \cdot d\mathbf{r}$ **is independent of path** if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ textbook

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} \; is\ independent\ of\ path\ if \int_{C_1} \vec{F} \cdot \mathrm{d}\vec{r} = \int_{C_1'} \vec{F} \cdot \mathrm{d}\vec{r}$$

used (to avoid confusion)

For any two paths C_1 and C'_1 in D that have the same initial point and the same terminal point, the line integrals alone C_1 and C'_1 are the same.

Can we find such a $\vec{F}$ whose line integral is independent of path?



16.3 Independence of Path

Independence of Path

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} \; is\ independent\ of\ path\ if \int_{C_1} \vec{F} \cdot \mathrm{d}\vec{r} = \int_{C_1'} \vec{F} \cdot \mathrm{d}\vec{r}$$

Can we find such a $\vec{F}$ whose line integral is independent of path?

Recall:

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

We know that the above integral depends only on the initial point and terminal point. Therefore, the integral is independent of path, i.e.,

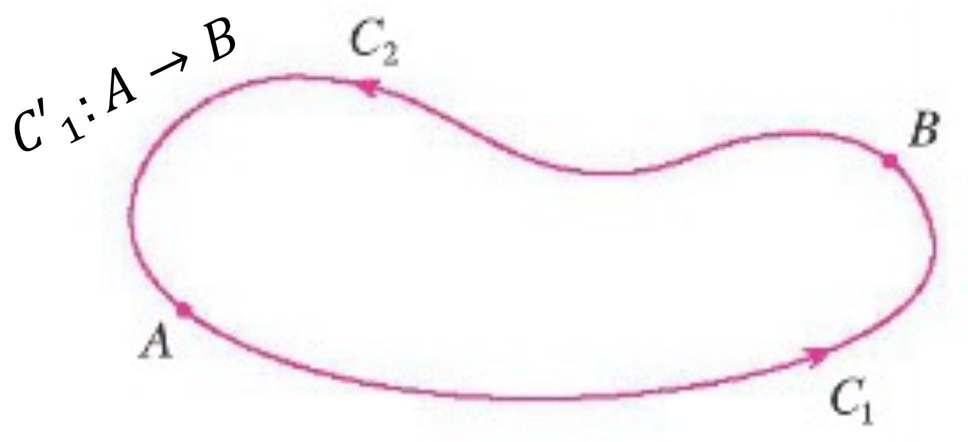
$$\int_{C_1} \vec{F} \cdot \mathrm{d}\vec{r} = \int_{C_1'} \vec{F} \cdot \mathrm{d}\vec{r}$$

$When\ \vec{F} = \nabla f, the\ line\ integral\ of\ \vec{F}$ is independent of path.

f is called a potential function.

Closed Curve and Path Independence

TBD



$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$$

$$C_2: B \to A$$

$$C'_1: A \to B$$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = -\int_{C'_1} \vec{F}\cdot d\vec{r}$$

The path independence yields:

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C'_1} \vec{F}\cdot d\vec{r}$$

We have:

$$\int_{C_1} \vec{F}\cdot d\vec{r} = -\int_{C_2} \vec{F}\cdot d\vec{r}$$

$$\int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = 0.$$

$$\oint \vec{F}\cdot d\vec{r} = 0$$

The line integral along a closed curve is zero.

What We Discussed or Will Discuss:

TBD

- Find the potential function f of $\vec{F} = (P, Q)$, i.e., $\vec{F} = \nabla f$, to simplify the line integral; $f_x = P$ *and* $f_y = Q$.
- Define the independence of path

$$\int_{C_1} \vec{F} \cdot \mathrm{d}\vec{r} = \int_{C_1'} \vec{F} \cdot \mathrm{d}\vec{r}$$

- Show the equivalent property:

$$\oint \vec{F} \cdot d\vec{r} = 0$$

- Define a "conservative" field using the above.
- Find a simple (math) expression to determine whether a field is conservative, $Q_x = P_y$ $(f_{yx} = f_{xy})$

$$\nabla \times \vec{F} = k\left(Q_x - P_y\right) = 0 \qquad \nabla \times \nabla f = 0$$

if and only if

Independence of paths
conservative

⟺

$\vec{F} = \nabla f$ on D

if and only if

⇕

⇓

$\oint \vec{F} \cdot d\vec{r} = 0$

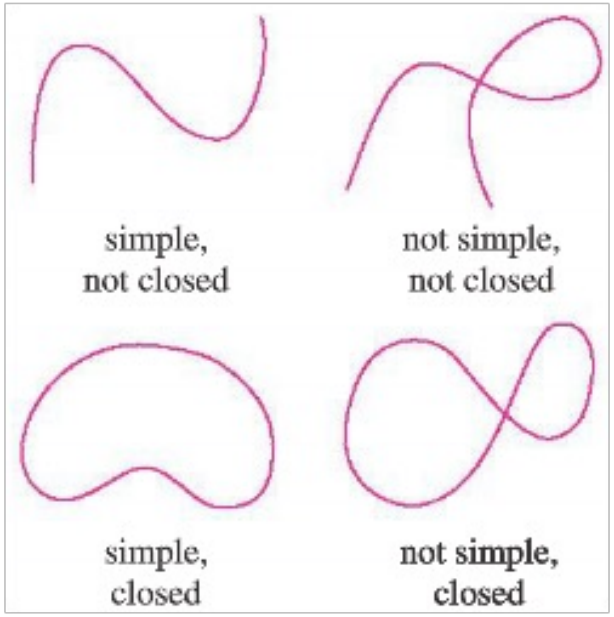
⇐

$\nabla \times \vec{F} = 0$ throughout D

$\nabla \times \nabla f = 0$

16.3 The Fundamental Theorem for Line Integrals

Simple curve: a curve that doesn't intersect itself anywhere between its endpoints.

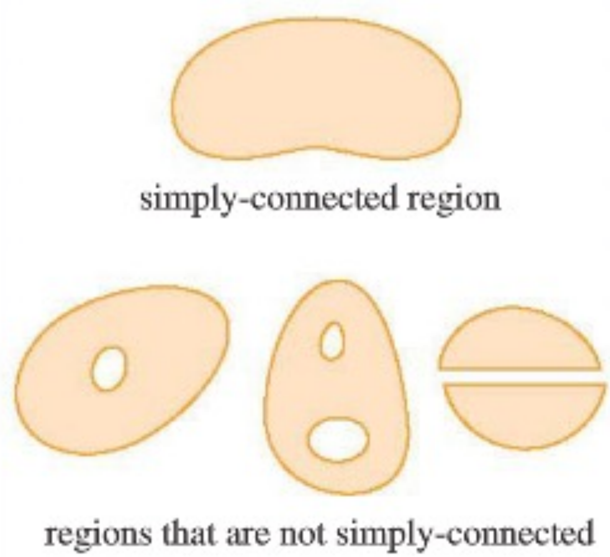


A simple closed curve:

$$\mathbf{r}(a) = \mathbf{r}(b)$$

$$\mathbf{r}(t_1) \neq \mathbf{r}(t_2) \text{ when } a < t_1 < t_2 < b.$$

Simply-connected region: a connected region D such that every simple closed curve in D encloses only points that are in D.



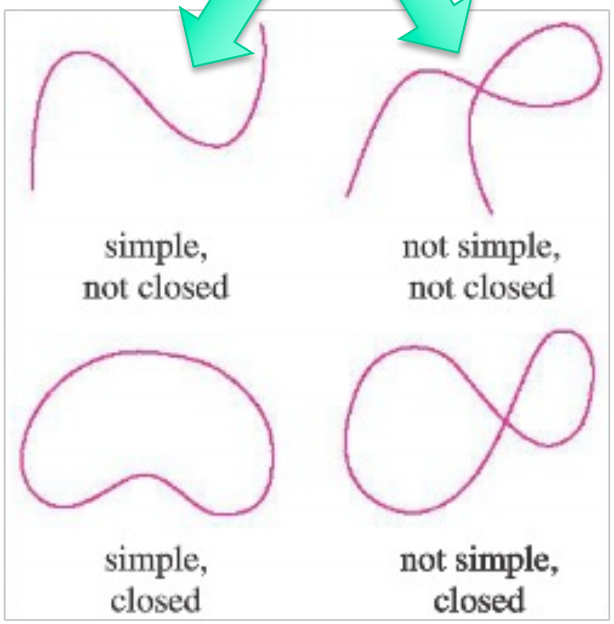
16.3 The Fundamental Theorem for Line Integrals

Simple curve: a curve that doesn't intersect itself anywhere between its endpoints

Simply-connected region: a connected region D such that every simple closed curve in D encloses only points that are in D

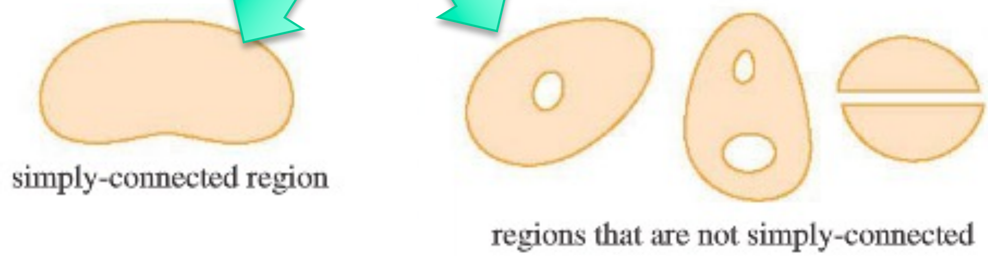
Intersection?

No Yes



singularity?

No Yes



simple (no intersection)

$\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ when $a < t_1 < t_2 < b$.

closed $\mathbf{r}(a) = \mathbf{r}(b)$

16.3 The Fundamental Theorem for Line Integrals TBD

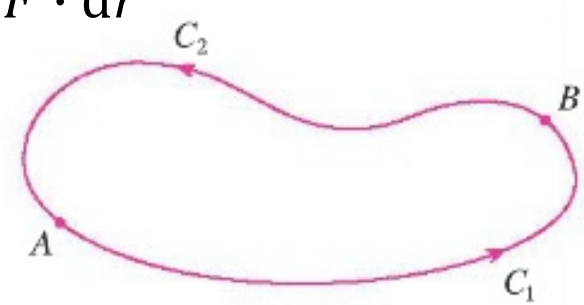
Independence of Path

3 Theorem $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

Definition $\int_C \vec{F} \cdot d\vec{r}$ *is independent of path if* $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1'} \vec{F} \cdot d\vec{r}$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{-C_2} \vec{F} \cdot d\vec{r}$$

$$= \int_{C1} \vec{F} \cdot d\vec{r} - \int_{C_1'} \vec{F} \cdot d\vec{r} = 0$$



A closed curve

Conversely,

$$0 = \int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{-C_2} \vec{F} \cdot d\vec{r}$$

$$= \int_{C1} \vec{F} \cdot d\vec{r} - \int_{C_1'} \vec{F} \cdot d\vec{r} \Rightarrow \int_{C1} \vec{F} \cdot d\vec{r} = \int_{C_1'} \vec{F} \cdot d\vec{r}$$ Independent

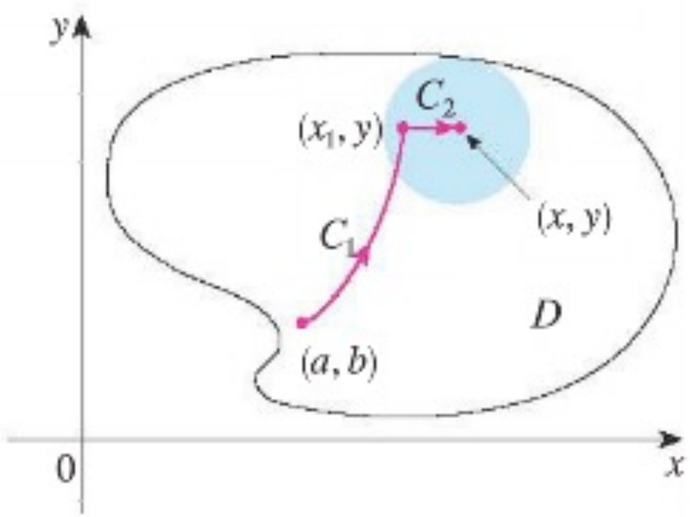
16.3: Conservative

Supp

4 Theorem Suppose **F** is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then **F** is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

PROOF Let $A(a, b)$ be a fixed point in D. We construct the desired potential function f by defining

$$\vec{F} = (P, Q) \qquad \vec{F} = \nabla f$$



$$P = \frac{\partial f}{\partial x} \qquad \text{and} \qquad Q = \frac{\partial f}{\partial y}$$

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial Q}{\partial x}$$

$$Q_x = P_y$$

$$\nabla \times \vec{F} = k\left(Q_x - P_y\right) = 0$$

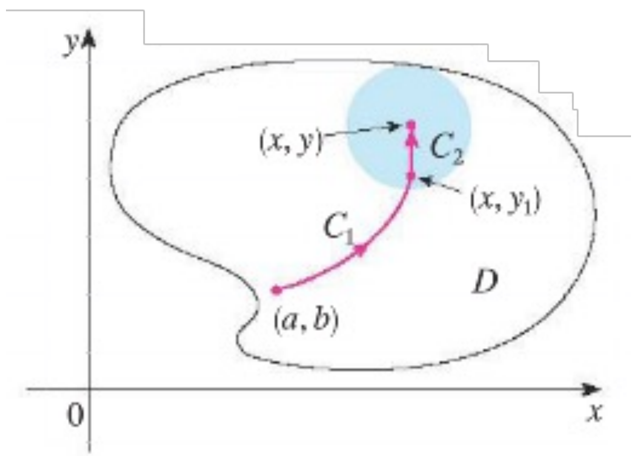
$\vec{F}$ is conservative

$$\nabla \times \nabla f = 0$$

16.3 Conservative Vector Field: A proof

Supp

$$f(x, y) = \int_{(a,b)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r} \qquad \mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$$



$$f(x, y) = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{(a,b)}^{(x_1,y)} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

$$\frac{\partial}{\partial x} f(x, y) = 0 + \frac{\partial}{\partial x} \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

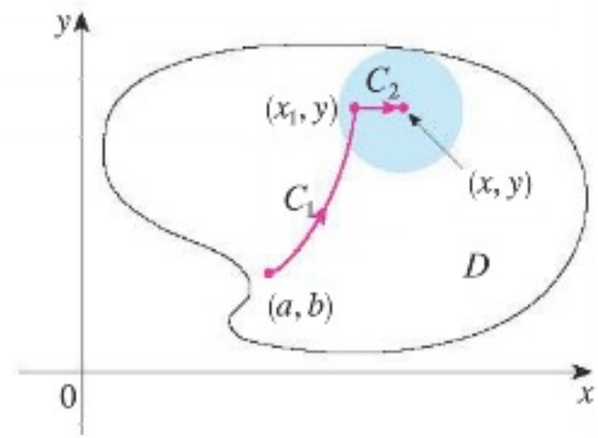
- Lower and upper limits are not a function of x
- Because of independence of path, the integral is not function of x.

$x_1 \approx t \approx x,$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} P\,dx + Q\,dy$$

$$\frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial x} \int_{C_2} P\,dx + Q\,dy = \frac{\partial}{\partial x} \int_{x_1}^{x} P(t, y)\,dt = P(x, y)$$

- dy=0 on C_2



Similarly

$$\frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y} \int_{C_2} P\,dx + Q\,dy = \frac{\partial}{\partial y} \int_{y_1}^{y} Q(x, t)\,dt = Q(x, y$$

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} = \nabla f$$

16.3 Conservative Vector Field

5 Theorem If $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\nabla \times \vec{F} = k\left(Q_x - P_y\right) = 0$$

6 Theorem Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

$$\nabla \times F = 0$$

$$recall:\ \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left(0, 0, Q_x - P_y\right)$$

CCC:
curl, cross product, conservative

16.3 The Fundamental Theorem for Line Integrals

6 Theorem Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

$$\nabla \times \vec{F} = k\left(Q_x - P_y\right) = 0$$

V EXAMPLE 2 Determine whether or not the vector field

$$\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + (x - 2)\,\mathbf{j}$$

is conservative.

$$(P, Q) = (x - y, x - 2)$$

$$P_y = -1 \qquad Q_x = 1$$

$$P_y \neq Q_x$$

V EXAMPLE 3 Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$$

is conservative.

$$P_y = 2x \qquad Q_x = 2x$$

$$P_y = Q_x$$

Revisit: Potential Function

EXAMPLE 5 If $\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + (2xy + e^{3z})\,\mathbf{j} + 3ye^{3z}\,\mathbf{k}$, find a function f such that $\nabla f = \mathbf{F}$.

$$\nabla f = (f_x, f_y, f_z) = (y^2, 2xy + e^{3z}, 3ye^{3z})$$

$$f_x = y^2 \qquad f_y = 2xy + e^{3z} \qquad f_z = 3ye^{3z}$$

$$f = xy^2 + g(y, z)$$

$$f_y = 2xy + g_y(y, z) \Rightarrow g_y(y, z) = e^{3z}$$

$$g = ye^{3z} + h(z)$$

$$f = xy^2 + ye^{3z} + h(z)$$

$$f_z = 3ye^{3z} + h_z(z) \Rightarrow h_z(z) = 0$$

$$h = K$$

$$f = xy^2 + ye^{3z} + K$$

16.3 Computing Line Integrals using a Potential Function

EXAMPLE 4

(a) If $\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.

(b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by

$$\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j} \qquad 0 \leqslant t \leqslant \pi$$

$P_y = 2x$

$Q_x = 2x$

$P_y = Q_x$

$$\nabla f = (f_x, f_y) = (3 + 2xy, x^2 - 3y^2)$$

$$f_x = 3 + 2xy$$

$$f = 3x + x^2 y + g(y)$$

$$f_y = x^2 + g_y$$

$$f_y = x^2 - 3y^2$$

$$g_y = -3y^2$$

$$g = -y^3 + K$$

$$f = 3x + x^2 y - y^3 + K$$

b at $t = \pi$

$\vec{r} = (0, -e^\pi)$

a at $t = 0$

$\vec{r} = (0,1)$

$$\int_C \vec{F} \cdot d\vec{r} = \int_c \nabla f \cdot d\vec{r} = \int_c df = f(b) - f(a)$$

$$= f(0, -e^\pi) - f(0,1) = e^{3\pi} + 1$$

A Journey into the Beauty of Vector Calculus

Math252, San Diego State University

Chapter 12

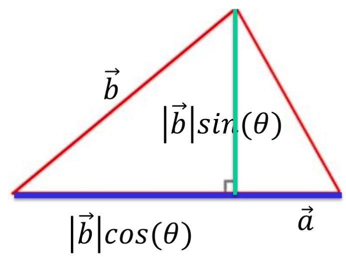
In Chapter 12, we discussed the following topics:

a) The Inner product, $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$

b) The Cross product, $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$

c) Equations of Lines, $\overrightarrow{PX} = t\vec{v}$

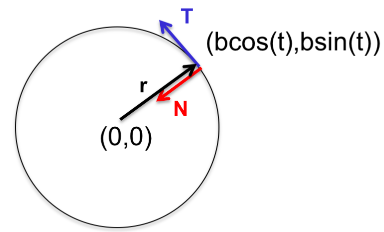
d) Equations of Plane. $\overrightarrow{PX} \cdot \vec{n} = 0$



$$projection = |\vec{b}|cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

$$height = |\vec{b}|sin(\theta) = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|}$$

Chapter 13



A circle: $x^2+y^2=b^2$

A parameterization of the circle:
r(t) = (bcos(t), bsin(t))

The derivative of the vector **r** is
r'(t) = (-bsin(t), bcos(t))

The magnitude of **r'(t)** is
|r'(t)|= $\sqrt{b^2 \sin^2(t) + b^2 \sin^2(t)} = b$

The unit tangent vector is

$$\vec{T} = \frac{\vec{r}'}{|\vec{r}'|} = (-\sin(t),\ \cos(t))$$

The derivative of the unit tangent vector is

$$\vec{T}' = \frac{d\vec{T}}{dt} = (-\cos(t),\ -\sin(t))$$

The unit normal vector is

$$\vec{N} = \frac{\vec{T}'}{|\vec{T}'|} = (-\cos(t),\ -\sin(t)) = \frac{-\vec{r}}{b}$$

The curvature is κ=|T'|/|r'|=1/b

Inner & cross products

$\vec{r} \cdot \vec{T} = 0$ $\quad \vec{T} \cdot \vec{N} = 0$ (orthogonal)

Chapter 14

One Formula Summary for Sections 14.4 – 14.8

(14.4) $df = f_x dx + f_y dy = \nabla f \cdot d\vec{r}$ — total differential

(14.5) $\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = \nabla f \cdot \frac{d\vec{r}}{dt}$ — chain rule with $x = x(t)$ and $y = y(t)$

(14.6) $D_u f = (f_x, f_y) \cdot (a, b) = \nabla f \cdot \frac{d\vec{r}}{|d\vec{r}|}$ — directional derivative $\vec{u} = \frac{\Delta\vec{r}}{|\Delta\vec{r}|}$

(14.7) $\nabla f = 0$ — max and min

(14.8) $\nabla f = \lambda \nabla g$ — Lagrange multiplier

Chapter 16

Formulas for Grad, Div, Curl, and the Laplacian

	Cartesian (x, y, z) **i**, **j**, and **k** are unit vectors in the directions of increasing x, y, and z. **P, Q, and R** are the scalar components of $\mathbf{F}(x, y, z)$ in these directions.
Gradient	$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$
Divergence	$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$
Curl	$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$
Laplacian	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

The Fundamental Theorem of Line Integrals

1. Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ be a vector field whose components are continuous throughout an open connected region D in space. Then there exists a differentiable function f such that
$$\mathbf{F} = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$
if and only if for all points A and B in D the value of $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path joining A to B in D.
2. If the integral is independent of the path from A to B, its value is
$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Green's Theorem (Tangential Form) $\iint_R \nabla \times \vec{F} \cdot \vec{k}\, dxdy = \oint \vec{F} \cdot d\vec{r}$

Stokes' Theorem $\iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$

Green's Theorem (Normal Form) $\iint_R \nabla \cdot \vec{F} dxdy = \oint \vec{F} \cdot \vec{n}\, ds$

Divergence Theorem $\iiint_E \text{div}\, \mathbf{F}\, dV = \iint \vec{F} \cdot \vec{n}\, dS$

Learning Outcomes

Formulas for Grad, Div, Curl, and the Laplacian

	Cartesian (x, y, z) $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ are unit vectors in the directions of increasing x, y, and z. **P, Q, and R** are the scalar components of $\mathbf{F}(x, y, z)$ in these directions.
Gradient	$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$
Divergence	$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$
Curl	$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$
Laplacian	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

The Fundamental Theorem of Line Integrals

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if and only if for all points A and B in D the value of $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path joining A to B in D.

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Green's Theorem (Normal Form) $\iint_R \nabla \cdot \vec{F}\, dxdy = \oint \vec{F} \cdot \vec{n}\, ds$

Divergence Theorem $\iiint_E \text{div}\, \mathbf{F}\, dV = \iint \vec{F} \cdot \vec{n}\, dS$

Green's Theorem

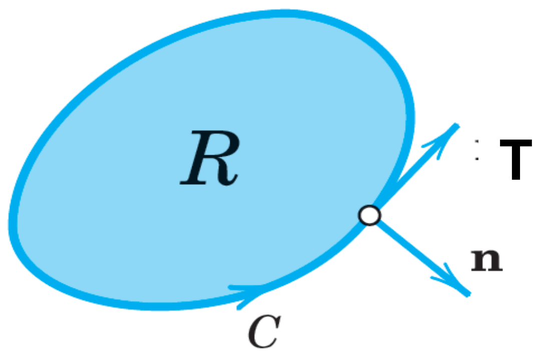
$$\begin{vmatrix} P & Q \\ dx & dy \end{vmatrix}$$

$$= Pdx + Qdy$$

$$\oint Pdx + Qdy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P(x,y) & Q(x,y) \end{vmatrix}$$

$$= Q_x - P_y$$



$$\begin{vmatrix} P & Q \\ dx & dy \end{vmatrix}$$

$$= Pdy - Qdx$$

$$\oint Pdy - Qdx = \iint \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dxdy$$

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P(x,y) & Q(x,y) \end{vmatrix}$$

$$= P_x + Q_y$$

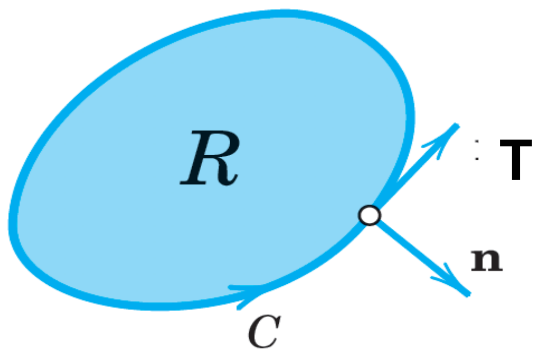
Green's Theorem

$$\oint \vec{F} \cdot d\vec{r} = \iint (\nabla \times \vec{F})\, dxdy$$

$$\oint P dx + Q dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

$$\begin{vmatrix} P & Q \\ dx & dy \end{vmatrix} = P dx + Q dy$$

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P(x,y) & Q(x,y) \end{vmatrix} = Q_x - P_y$$



16.3 An Example

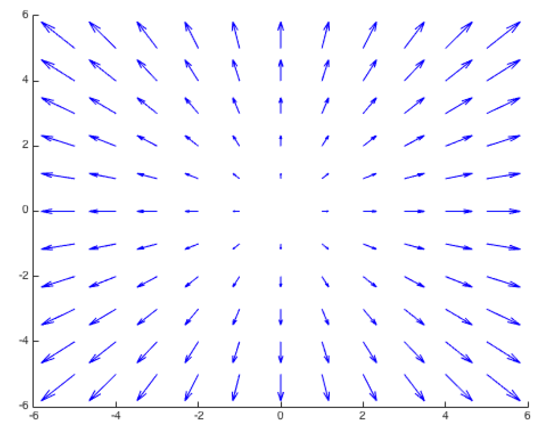
Example

Let C be the closed curve described by C: $x^2 + y^2 = a^2$, and 2: $\vec{F} = \left(\frac{x}{2}, \frac{y}{2}\right)$
Evaluate the line integral of F along C.

Key Points:

$\nabla \times \vec{F} = k(Q_x - P_y) = 0$ throughout D

2. Uniform Expansion Field



F is conservative. $\vec{F} = \nabla f$

$f = \frac{1}{4}(x^2 + y^2) + C$

$\oint \vec{F} \cdot d\vec{r} = 0$ independence of path

Independence of paths
conservative

$\Longleftrightarrow$ $\vec{F} = \nabla f$ on D

$\Updownarrow$ $\Downarrow$

$$\oint \vec{F} \cdot d\vec{r} = 0$$

$\Longleftarrow$ $\nabla \times \vec{F} = 0$ throughout D

$If\ \vec{F} = \nabla f, \vec{F} = (f_x, f_y),$

we have $\nabla \times \vec{F} = c(Q_x - P_y) = k(f_{yx} - f_{xy}) = 0$

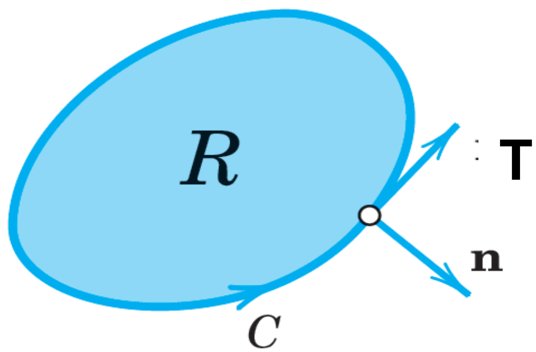
Green's Theorem

$$\oint \vec{F} \cdot d\vec{r} = \iint (\nabla \times \vec{F})\,dxdy$$

$$\oint Pdx + Qdy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

$$\begin{vmatrix} P & Q \\ dx & dy \end{vmatrix} = Pdx + Qdy$$

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P(x,y) & Q(x,y) \end{vmatrix} = Q_x - P_y$$

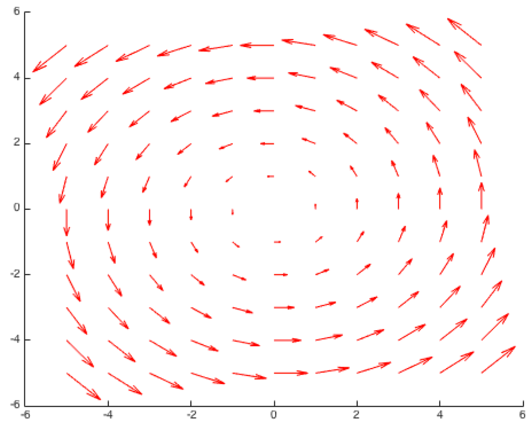


$$\vec{F} = \left(\frac{x}{2}, \frac{y}{2}\right)$$

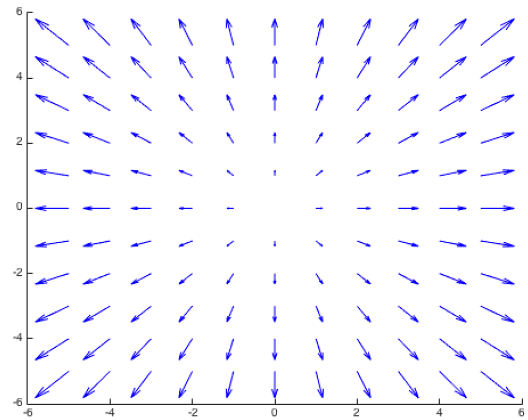
$$\nabla \times \vec{F} = k\left(Q_x - P_y\right) = 0$$

Four Vector Fields

1. Uniform Rotation Field



2. Uniform Expansion Field

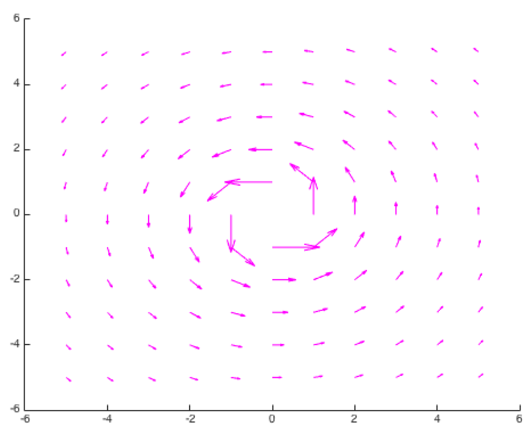


$$1:\ \vec{F} = \left(\frac{-y}{2}, \frac{x}{2}\right)$$

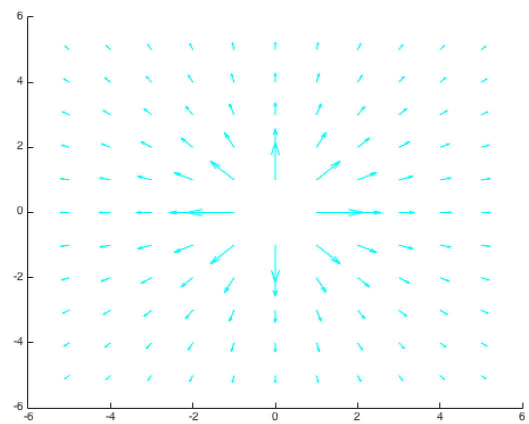
$$2:\ \vec{F} = \left(\frac{x}{2}, \frac{y}{2}\right)$$

$$\left[\vec{F} = \nabla f;\ f = \frac{1}{4}\left(x^2 + y^2\right)\right]$$

3. Whirlpool Field



4. 2D Electrical Field



$$3:\ \vec{F} = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right)$$

$$4:\ \vec{F} = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right)$$

$$\left[\vec{F} = \nabla f;\ f = \ln\sqrt{x^2 + y^2}\right]$$

16.3 A Summary

2 Theorem Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \nabla f \cdot d\vec{r} = \int_a^b df = f(b) - f(a).$$

Independence of Path $\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$

3 Theorem $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

F is conservative. $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ throughout D

$$2:\ \vec{F} = \left(\frac{x}{2}, \frac{y}{2}\right) \qquad C:\ x^2 + y^2 = a^2 \qquad \oint \vec{F} \cdot d\vec{r} = 0$$