MATH 252 --- Calculus III

Spring 2025

Instructor: Bo-Wen Shen, Ph.D.

Lecture #26:
Sections 16.4
Green's Theorem

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Learning Outcomes

Formulas for Grad, Div, Curl, and the Laplacian

	Cartesian (x, y, z) $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors in the directions of increasing x, y, and z. P, Q, and R are the scalar components of $\mathbf{F}(x, y, z)$ in these directions.
Gradient	$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$
Divergence	$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$
Curl	$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$
Laplacian	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

The Fundamental Theorem of Line Integrals

1. Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ be a vector field whose components are continuous throughout an open connected region D in space. Then there exists a differentiable function f such that

$$\mathbf{F} = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

if and only if for all points A and B in D the value of $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path joining A to B in D.

2. If the integral is independent of the path from A to B, its value is

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Green's Theorem (Tangential Form) $\quad \iint_R \nabla \times \vec{F} \cdot \vec{k}\, dxdy = \oint \vec{F} \cdot d\vec{r}$

Stokes' Theorem $\quad \iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$

Green's Theorem (Normal Form) $\quad \iint_R \nabla \cdot \vec{F}\, dxdy = \oint \vec{F} \cdot \vec{n}\, ds$

Divergence Theorem $\quad \iiint_E \text{div}\, \mathbf{F}\, dV = \iint \vec{F} \cdot \vec{n}\, dS$

Green's Theorem

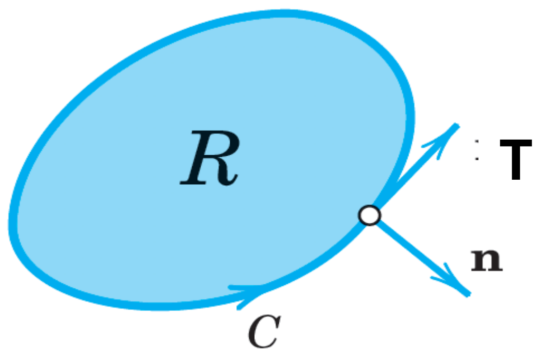
$$\begin{vmatrix} P & Q \\ dx & dy \end{vmatrix}$$

$$= Pdx + Qdy$$

$$\oint Pdx + Qdy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

$$\begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ P(x,y) & Q(x,y) \end{vmatrix}$$

$$= Q_x - P_y$$



$$\begin{vmatrix} P & Q \\ dx & dy \end{vmatrix}$$

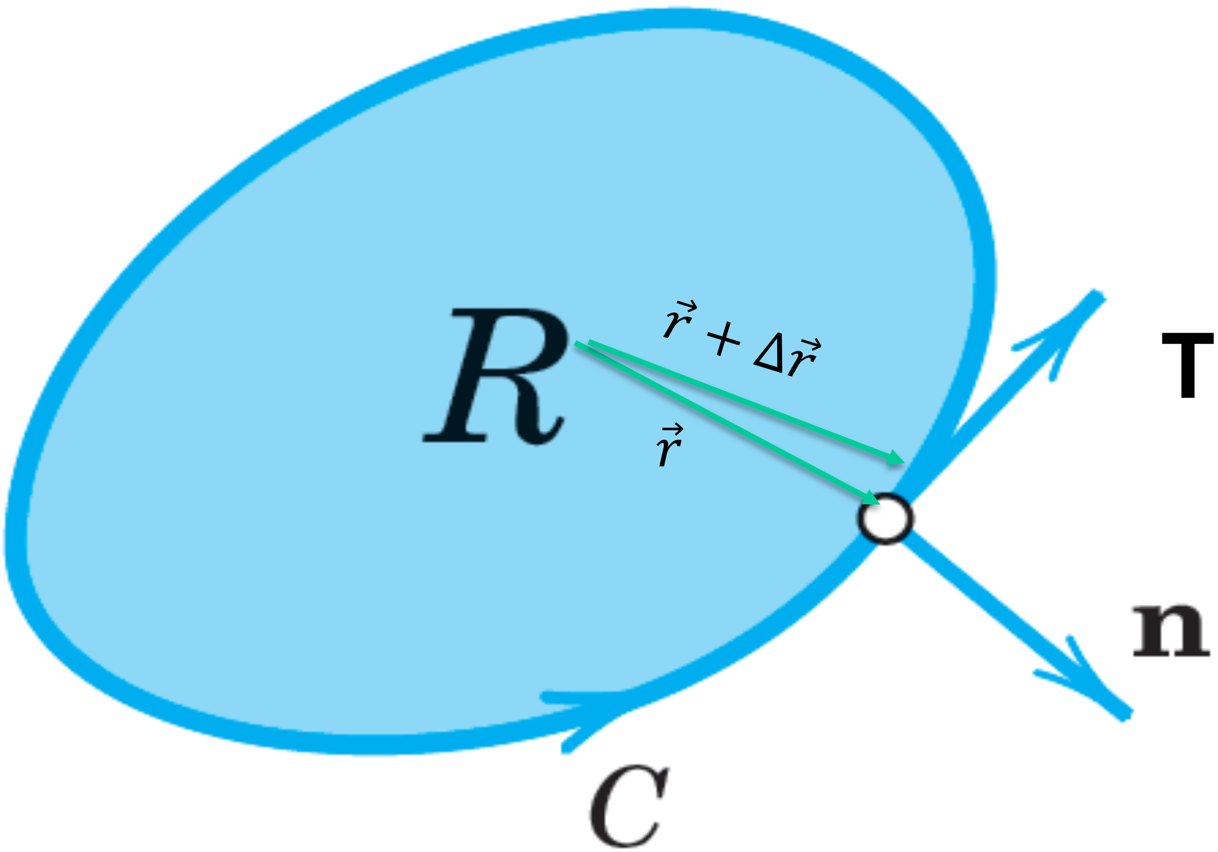
$$= Pdy - Qdx$$

$$\oint Pdy - Qdx = \iint \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dxdy$$

$$\begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ P(x,y) & Q(x,y) \end{vmatrix}$$

$$= P_x + Q_y$$

$$\vec{T} = \frac{\vec{r}'}{|\vec{r}'|} \approx (\vec{r} + \Delta\vec{r}) - \vec{r}$$



Topics to be discussed

potential function ← line integral → double integral

$$f(b) - f(a) = \int_a^b \nabla f \cdot \mathrm{d}\vec{r} = \boxed{\int_C \vec{F} \cdot \vec{T} ds}$$

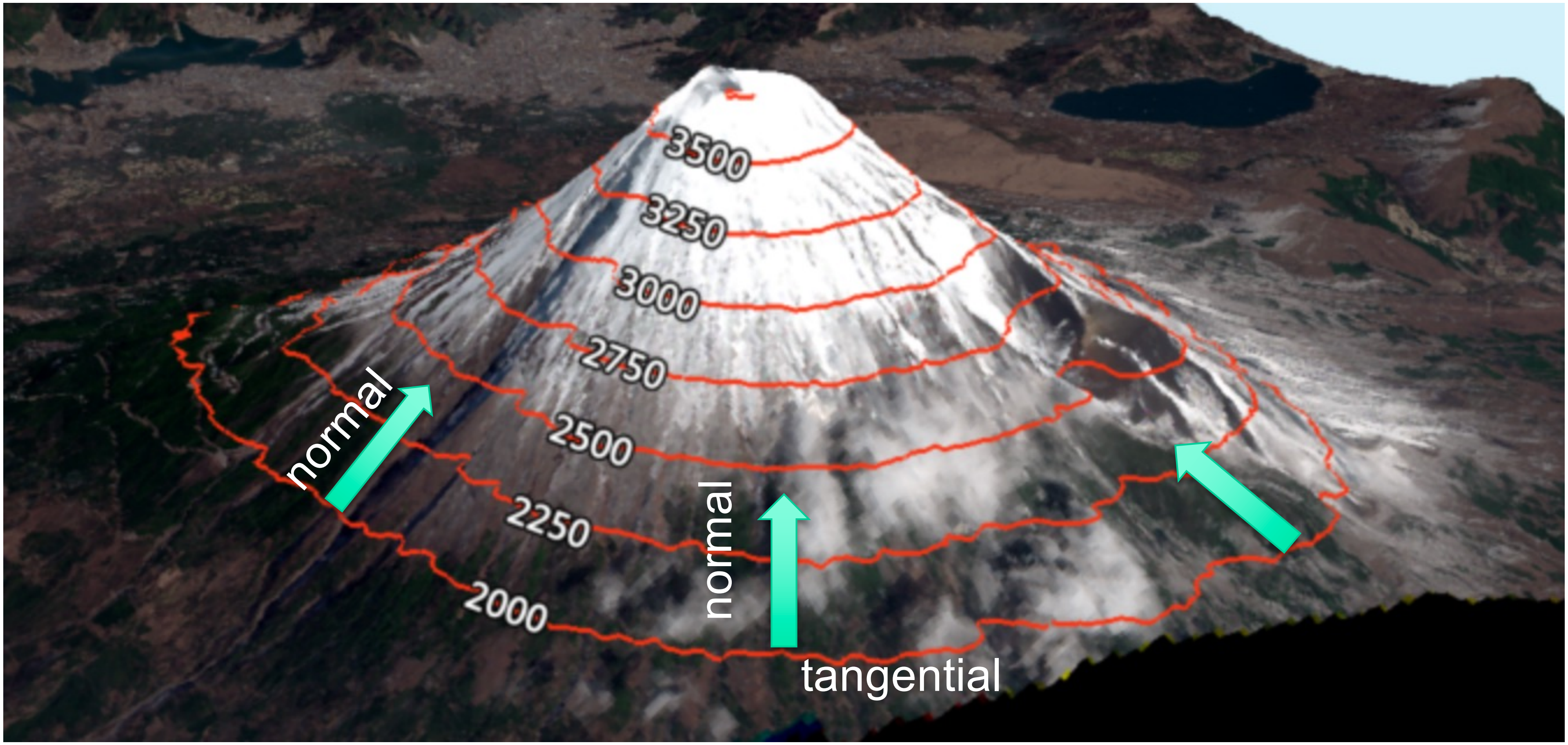
A closed curve

$$0 = \oint df = \oint \nabla f \cdot d\vec{r} = \oint \vec{F} \cdot \vec{T}\, ds$$

$$\oint \vec{F} \cdot \vec{T}\, ds \quad = \iint \nabla \times \vec{F} \cdot \vec{k}\, dxdy$$

zero order ← 1st order → 2nd order

Tangential vs. Normal



Green's Theorem (tangential form)

circulation (or work) = double integral of "curl"

$$\begin{vmatrix} P & Q \\ dx & dy \end{vmatrix}$$

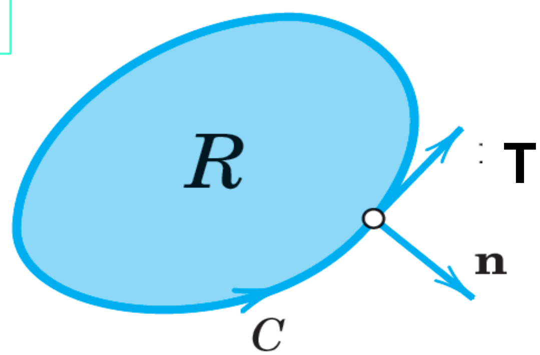
$$\oint P dx + Q dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P(x,y) & Q(x,y) \end{vmatrix}$$

$$= Q_x - P_y$$

$$\boxed{= P dx + Q dy = (P, Q) \cdot d\vec{r}}$$

$$= (P, Q) \cdot (dx, dy)$$



$$\vec{T} = \frac{d\vec{r}}{|d\vec{\boldsymbol{r}}|} = \frac{(dx, dy)}{ds}$$

$$(dx, dy) = \vec{T} ds$$

$$\boxed{= (P, Q) \cdot \vec{T} ds}$$

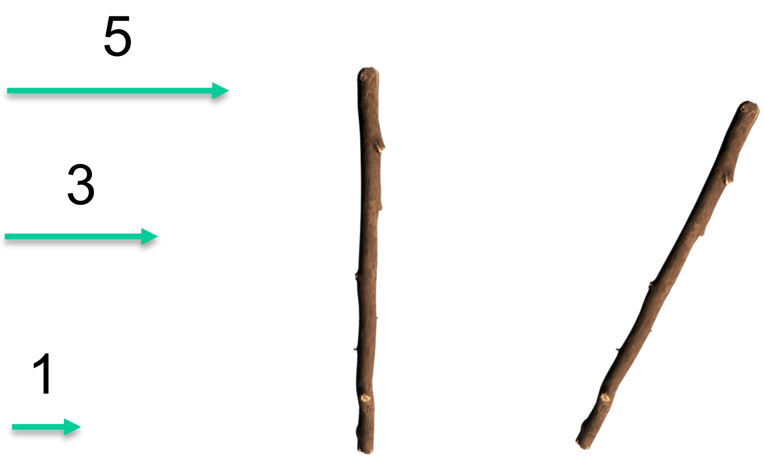
Curl & Divergence (2D Version)

Curl:

"a **C**ross product of ∇ and F"

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix}$$

$$\nabla \times \vec{F} = k\left(Q_x - P_y\right)$$



Green's Theorem (tangential form)

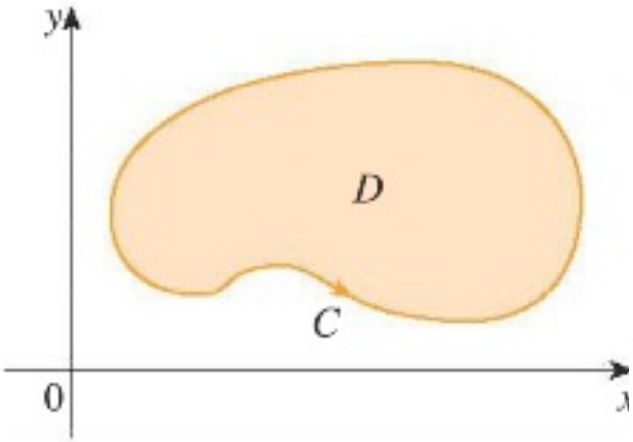
$$\oint \vec{F} \cdot \vec{T}\, ds = \oint P dx + Q dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Tangential
Form

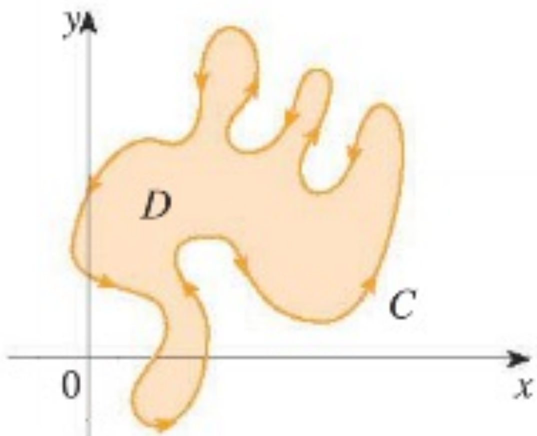
circulation $\iint (curl)$

Tom C.

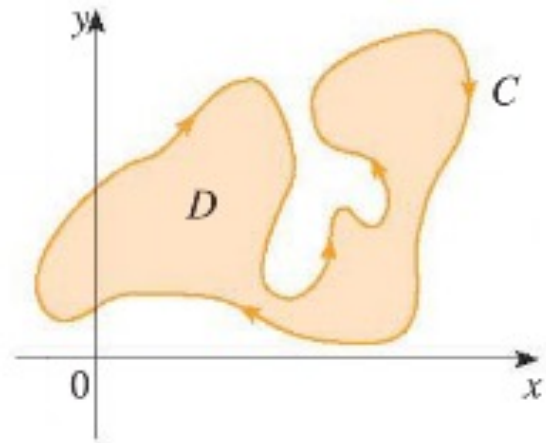
16.4 Green’s Theorem: Positive Orientation



Counterclockwise orientation



(a) Positive orientation

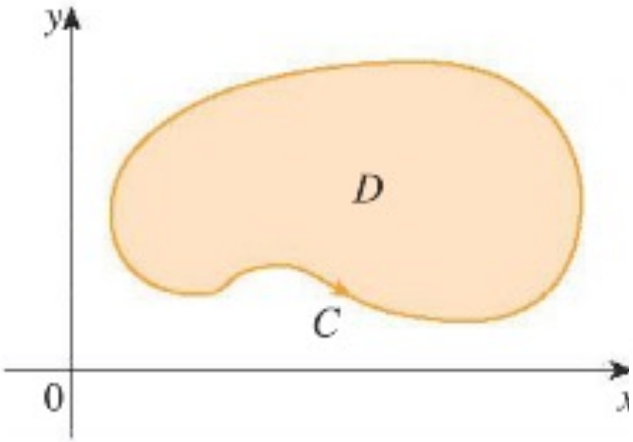


(b) Negative orientation

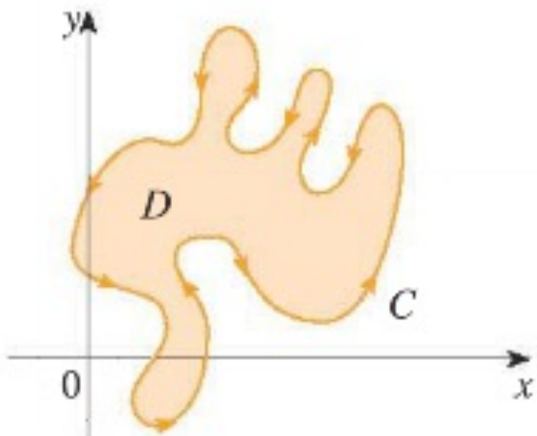
The curve is oriented so that the region D is always on the left as the curve is traversed.

➔ Along the entire boundary C of D in such a sense that D is on the left as we advance in the direction of integration

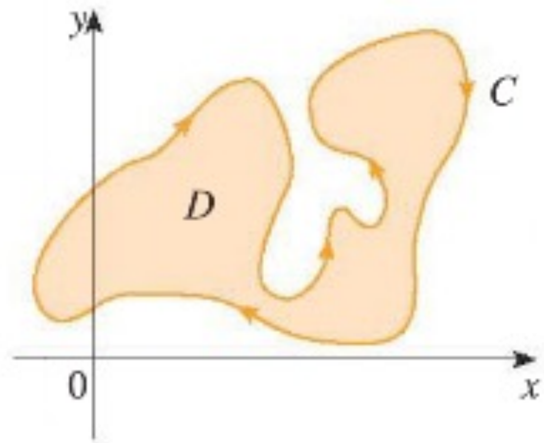
16.4 Green's Theorem



Counterclockwise orientation



(a) Positive orientation



(b) Negative orientation

The curve is oriented so that the region D is always on the left as the curve is traversed.

Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

circulation (or work) = double integral of "curl"

16.4 Green's Theorem

Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

∂D represents the positively oriented boundary curve of D

1 $$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{\partial D} P\,dx + Q\,dy$$

Double integral <--> Line integral

Notation $$\oint_C P\,dx + Q\,dy \quad \text{or} \quad \boxed{\oint_C} P\,dx + Q\,dy$$

$$\boxed{\oint_C}$$

Recall: $$\int_a^b F'(x)\,dx = F(b) - F(a)$$

integral <--> function evaluation

16.4 Green's Theorem (A proof)

Supp

PROOF OF GREEN'S THEOREM FOR THE CASE IN WHICH D IS A SIMPLE REGION

Goals

$$\boxed{\int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA} \qquad \int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA$$

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

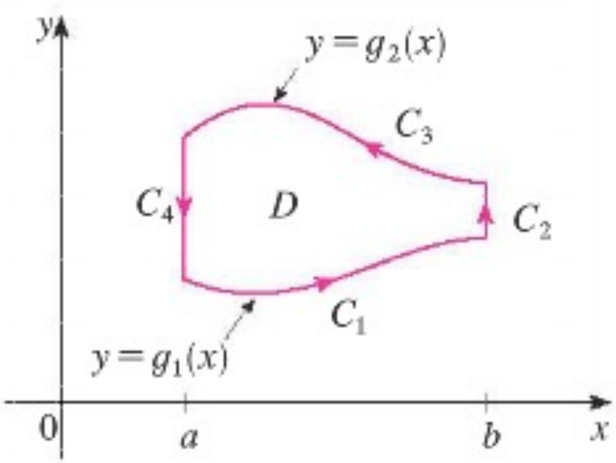
RHS

4

$$\boxed{\iint_D \frac{\partial P}{\partial y}\,dA} = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x, y)\,dy\,dx = \boxed{\int_a^b [P(x, g_2(x)) - P(x, g_1(x))]\,dx}$$

LHS

$$\int_C P(x, y)\,dx = \int_{C_1} P(x, y)\,dx + \int_{C_2} P(x, y)\,dx + \int_{C_3} P(x, y)\,dx + \int_{C_4} P(x, y)\,dx$$



$$\int_{C_1} P(x, y)\,dx = \boxed{\int_a^b P(x, g_1(x))\,dx}$$

$$\int_{C_3} P(x, y)\,dx = -\int_{-C_3} P(x, y)\,dx = \boxed{-\int_a^b P(x, g_2(x))\,dx}$$

$$\boxed{\int_{C_2} P(x, y)\,dx = 0 = \int_{C_4} P(x, y)\,dx}$$

$$\boxed{\int_C P(x, y)\,dx} = \boxed{\int_a^b P(x, g_1(x))\,dx - \int_a^b P(x, g_2(x))\,dx}$$

16.4 Green's Theorem

$$\oint \vec{F} \cdot \vec{T}\, ds = \oint \vec{F} \cdot d\vec{r}$$

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

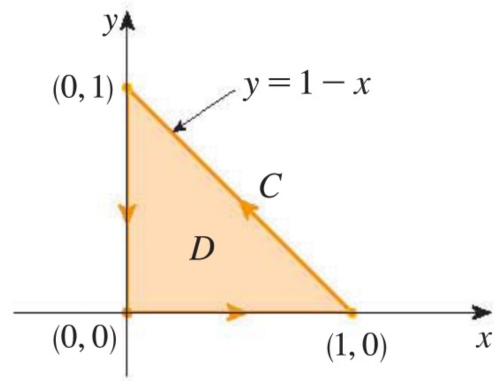


FIGURE 4

EXAMPLE 1 Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from (0, 0) to (1, 0), from (1, 0) to (0, 1), and from (0, 1) to (0, 0).

V EXAMPLE 2 Evaluate $\oint_C (3y - e^{\sin x})\,dx + \left(7x + \sqrt{y^4 + 1}\right) dy$, where C is the circle $x^2 + y^2 = 9$.

16.4 Green's Theorem: 1D Integral → 2D Integral

$$\oint \vec{F}\cdot\vec{T}\,ds = \oint \vec{F}\cdot d\vec{r} \qquad\qquad \int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

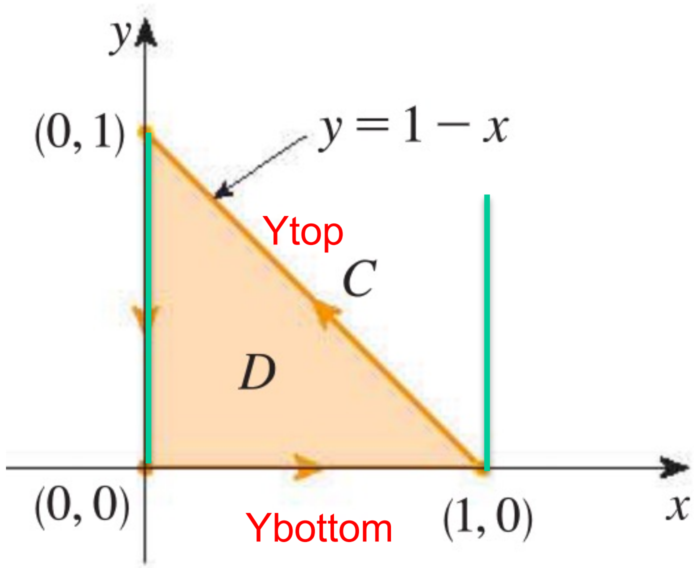
EXAMPLE 1 Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from $(0, 0)$ to $(1, 0)$, from $(1, 0)$ to $(0, 1)$, and from $(0, 1)$ to $(0, 0)$.

$$\oint \vec{F}\cdot d\vec{r} = \oint (x^4, xy)\cdot(dx, dy) \qquad\qquad (P, Q) = (x^4, xy)$$

$$\boxed{\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint (y-0)\,dxdy}$$

$$= \int_0^1 \int_0^{1-x} (y)\,dy\,dx = \int_0^1 \left[\frac{y^2}{2}\right]_0^{1-x} dx$$

$$= \int_0^1 \frac{(1-x)^2}{2}\,dx = \frac{1}{2}\left[\frac{x^3}{3} - x^2 + x\right]_0^1 = \frac{1}{6}$$



Example 2

V EXAMPLE 2 Evaluate $\oint_C (3y - e^{\sin x})\,dx + \left(7x + \sqrt{y^4 + 1}\right) dy$, where C is the circle $x^2 + y^2 = 9$.

$$\oint \vec{F} \cdot d\vec{r} = \oint \left(3y - e^{\sin x}, 7x + \sqrt{y^4 + 1}\right) \cdot (dx, dy)$$

$$(P, Q) = \left(3y - e^{\sin x}, 7x + \sqrt{y^4 + 1}\right)$$

$$\boxed{\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint (7 - 3) dxdy}$$

$$= 4 \iint dxdy \quad = 4 \iint dA \quad = 4(\pi r^2) = 36\pi$$

16.4 An Application to Compute an Area

Consider the Green's theorem:
$$\boxed{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy}$$

To compute an area, we can choose $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ $\quad LHS = \iint dA = A$

The following thee cases meet the above condition, i.e., $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$

$P(x, y) = 0$ $\quad$ $P(x, y) = -y$ $\quad$ $\boxed{P(x, y) = -\tfrac{1}{2}y}$ $\quad$ vector field

$Q(x, y) = x$ $\quad$ $Q(x, y) = 0$ $\quad$ $\boxed{Q(x, y) = \tfrac{1}{2}x}$

Plug either one of the above into the RHS

LHS

$$\oint_C x\,dy = -\oint_C y\,dx = \boxed{\tfrac{1}{2}\oint_C x\,dy - y\,dx} = \iint_D 1dA = A$$

Qdy $\quad$ Pdx $\quad$ TBD

16.4 An application to compute an area

EXAMPLE 3 Find the area enclosed by *the* circle $\frac{x^2}{a^2}+\frac{y^2}{a^2}=1$ by computing a line integral.

$$\iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$$

$$P(x,y) = -\tfrac{1}{2}y \qquad Q(x,y) = \tfrac{1}{2}x$$

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y} = 1$$

$$LHS = \iint_D 1dA = A$$

$$(x,y) = (a\cos(t), a\sin(t))$$

$$(dx,dy) = (-a\sin(t), a\cos(t))dt$$

$$RHS = \oint Pdx + Qdy = \frac{1}{2}\oint xdy - ydx$$

$$= \frac{1}{2}\int_0^{2\pi} a^2\cos^2(t)\,dt + a^2\sin^2(t)\,dt \;= \frac{1}{2}\int_0^{2\pi} a^2 dt = \boxed{\pi a^2}$$

EXAMPLE 3 Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Why Line Integral?

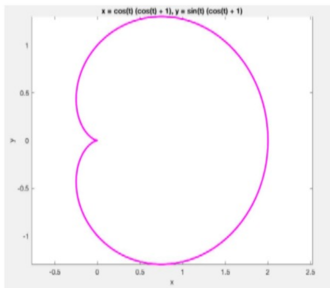
TBD

[10 points] Compute the area enclosed by the curve for an idealized heart C: r=1+cos(θ),

(a) ans = ____ .

(b) Your question ID is: 2987−6577−2499−119−*bshen@sdsu*. Please provide the security code: ____ .

(c) Provide derivations. [8 points]



(Left) Fifteen airplanes saluted URMC, healthcare providers, first responders, and essential workers in the Thomaston area by creating a heart above the community. Photo by Stacy Haygood. (Right) An idealized heart with r=1+cos(θ).

Green's Theorem (Tangential Form)

$$\iint_R \nabla \times \vec{F} \cdot \vec{k}\,dxdy = \oint \vec{F} \cdot d\vec{r}$$

Consider the line integral $\int_C x\,dy$.

- The above suggests that the area can be determined by the time evolution of the airplane's locations,(x(t), (y(t)).
- It can be done in parallel during the flights of 15 airplanes.

Double Integral

TBD

[10 points] Apply the following integral to compute the area enclosed by the curve for an idealized heart C: $r=1+\cos(\theta)$:

Ans = $\int_a^b \int_c^{1+\cos(\theta} f(r,\theta)\, r dr\, d\theta$.

$f(r,\theta)$ = [ ]

a= [ ] ; b = [ ] ;

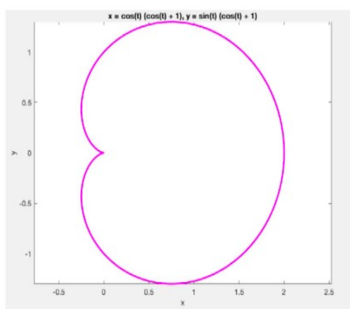
c= [ ] ;

Ans = [ ]

```
syms theta r rmax
% r = 1 + cos(theta) %not defined

rmax= 1 + cos(theta)
int(int(r, r, 0, rmax), theta, 0, 2*pi)
```

Your question ID is: 9339−1429−5629−3900−*bshen@sdsu*. Please provide the security code: [ ] .



```
>> int(int(r, r, 0, rmax), theta, 0, 2*pi)

ans =

(3*pi)/2
```

(Left) Fifteen airplanes saluted URMC, healthcare providers, first responders, and essential workers in the Thomaston area by creating a heart above the community. Photo by Stacy Haygood. (Right) An idealized heart with $r=1+\cos(\theta)$.

$\theta \in [-\pi, \pi]$ $\qquad$ $r \in [0, 1+\cos(\theta)]$

Line Integral

TBD

$$\iint dxdy = \iint rdrd\theta \quad = \int x\, dy$$

Green's Theorem:
double integral and line integral

$$\int x\, dy = \int (\cos(\theta) + \cos^2(\theta)) * (\cos(\theta) + \cos^2(\theta) - \sin^2(\theta))\, d\theta$$

syms x y theta

r = 1 + cos(theta)
x = r * cos(theta)
y = r * sin(theta)

int((cos(theta) + cos(theta)^2) * (cos(theta) + cos(theta)^2 - sin(theta)^2), theta, 0, 2*pi)

```
>> int( (cos(theta) + cos(theta)^2 ) * (cos(theta) + cos(theta)^2 - sin(theta)^2), theta, 0, 2*pi)

ans =

(3*pi)/2
```

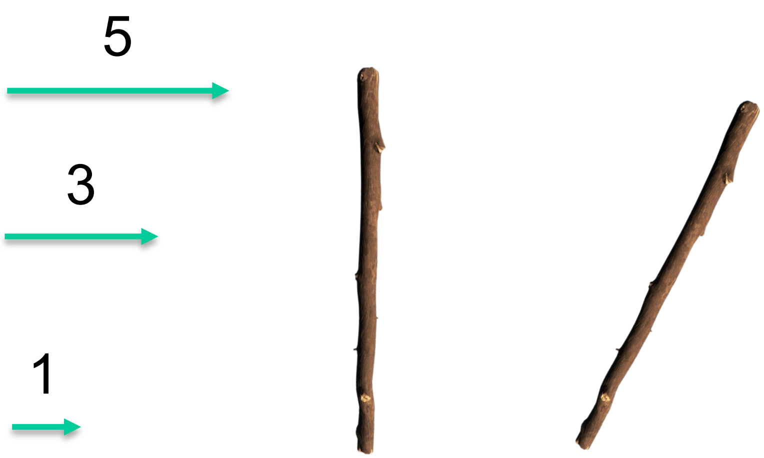
Curl & Divergence (2D Version)

Curl:

“a Cross product of ∇ and F”

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix}$$

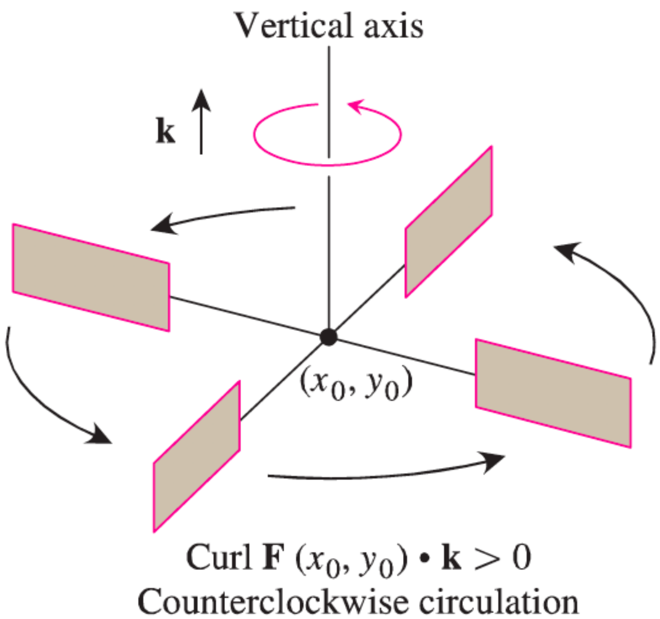
$$\nabla \times \vec{F} = k\left(Q_x - P_y\right)$$

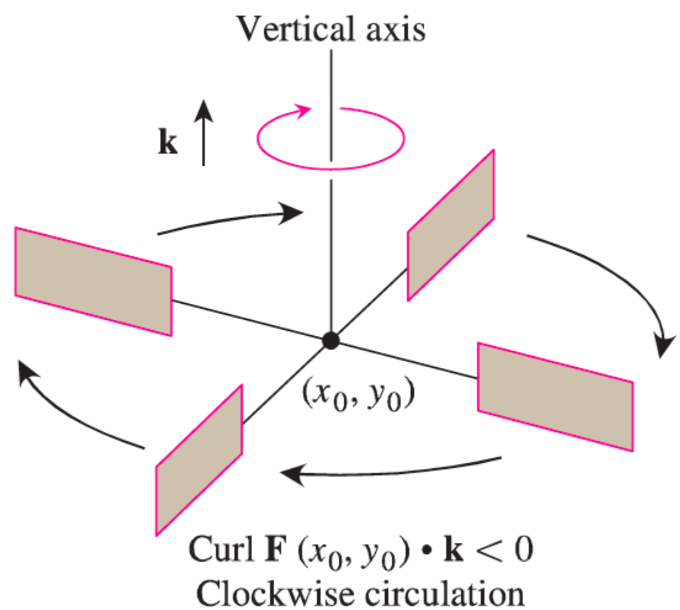


Rotation Field

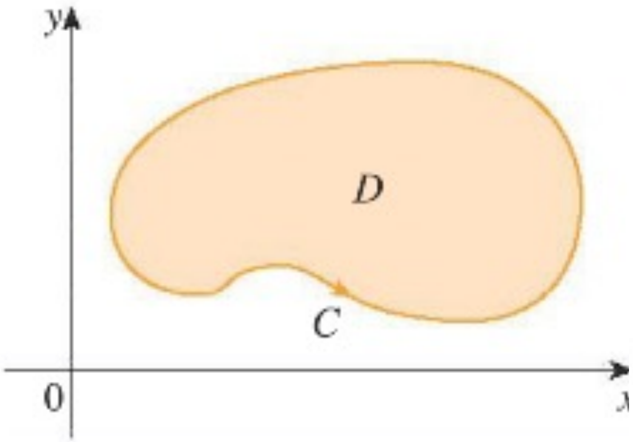
- Counter-clockwise Circulation
- $\nabla\times\vec{F}\cdot\boldsymbol{k} > 0$

- Clockwise Circulation
- $\nabla\times\vec{F}\cdot\boldsymbol{k} < 0$

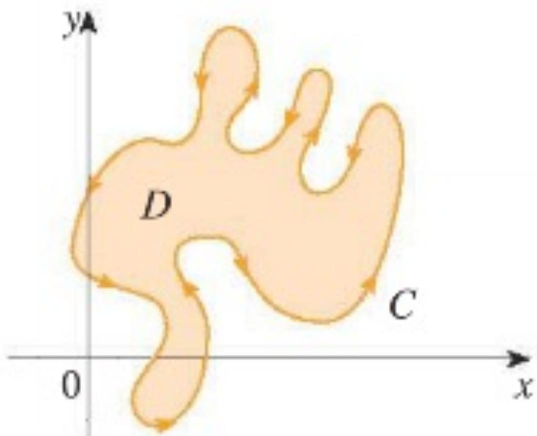




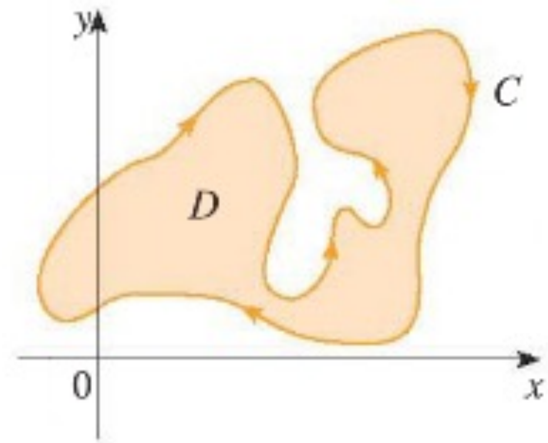
16.4 Green's Theorem (Vector Form)



Counterclockwise orientation



(a) Positive orientation



(b) Negative orientation

The curve is oriented so that the region D is always on the left as the curve is traversed.

➔ Along the entire boundary C of D in such a sense that D is on the left as we advance in the direction of integration

$$\oint P dx + Q dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

vector form

$$\oint \vec{F} \cdot \vec{T}\, ds \quad = \iint \nabla \times \vec{F} \cdot \vec{k}\, dxdy$$

16.4 Example 4

V EXAMPLE 4 Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

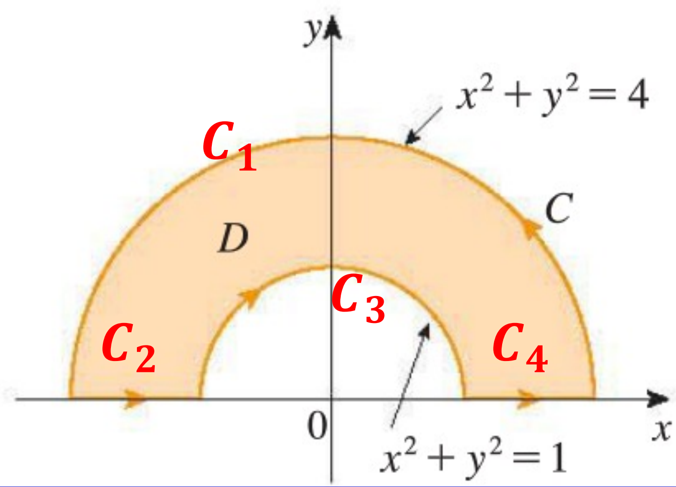
$$\oint \vec{F}\cdot\vec{T}\,ds \quad = \iint \nabla\times\vec{F}\cdot\vec{k}\,dxdy$$

$$\oint \vec{F}\cdot d\vec{r} = \oint (y^2, 3xy)\cdot(dx, dy)$$

The curve is oriented so that the region D is always on the left as the curve is traversed.

$$(P, Q) = (y^2, 3xy)$$

$$\boxed{\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint (3y - 2y)dxdy}$$



Example 4: Double Integral (.continued)

V EXAMPLE 4 Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

$$\boxed{\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint (3y - 2y)dxdy}$$

$$= \iint ydA \; = \int_0^{\pi}\int_1^{2} r\sin(\theta) r dr d\theta$$

$$= \int_0^{\pi}\left[\frac{r^3}{3}\right]_1^2 \sin(\theta)d\theta \; = -\frac{7}{3}[\cos(\theta)]_0^{\pi}$$

$$= \frac{14}{3}$$

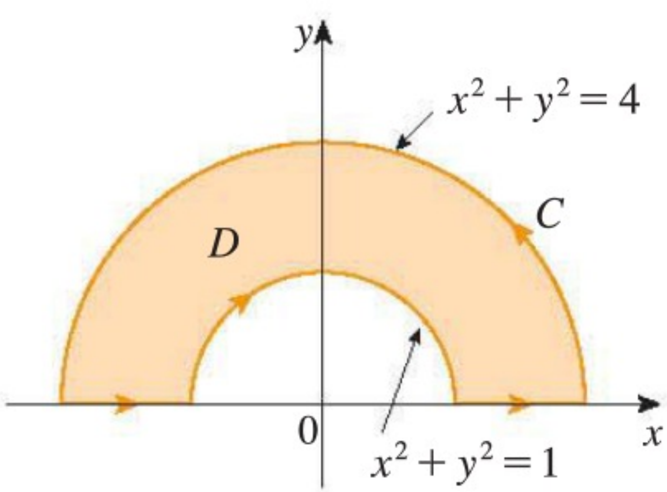


FIGURE 8

Example 4: Line Integrals (.continued)

V EXAMPLE 4 Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

$$\oint \vec{F} \cdot d\vec{r} = \int_{C_1+C_2+C_3+C_4} \vec{F} \cdot d\vec{r}$$

The curve is oriented so that the region D is always on the left as the curve is traversed.

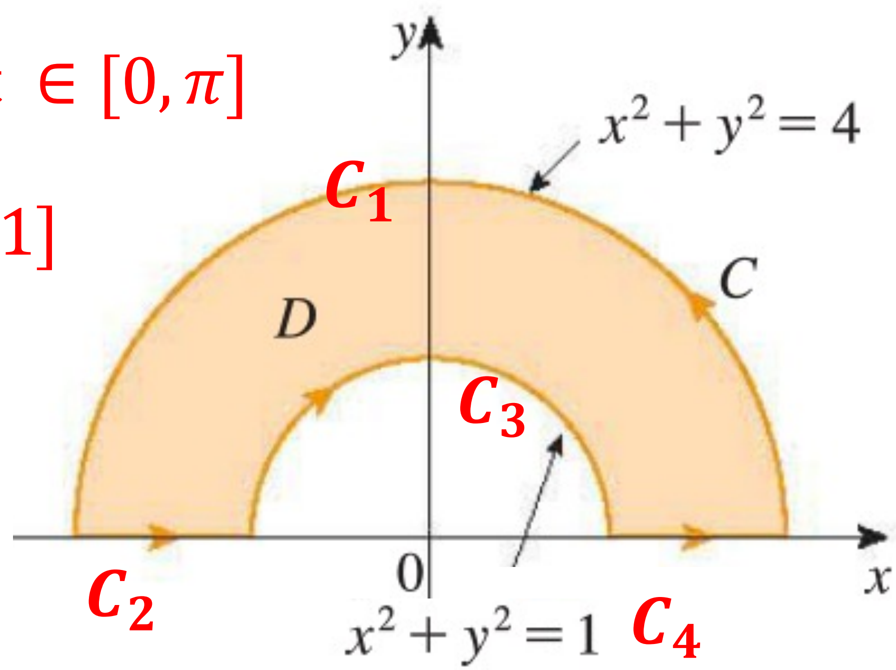
$C_1: (x, y) = (2cos(t), 2sin(t)) \quad t \in [0, \pi]$

$C_2: (x, y) = (t, 0) \quad t \in [-2, -1]$

$C_3: (x, y) = (cos(t), sin(t))$

$t \in [\pi, 0]$

$C_4: (x, y) = (t, 0) \quad t \in [1,2]$



Example 4: Line Integrals (.continued)

V EXAMPLE 4 Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

$$\oint \vec{F}\cdot d\vec{r} = \int_{C_1+C_2+C_3+C_4} \vec{F}\cdot d\vec{r} = \frac{14}{3}$$

$$\int_{C_i} \vec{F}\cdot d\vec{r} =$$

$C_1: (x,y) = (2cos(t), 2sin(t))$ $\quad$ $t \in [0,\pi]$ $\quad$ $-16 + \frac{64}{3}$

$C_2: (x,y) = (t,0)$ $\quad$ $t \in [-2,-1]$ $\quad$ 0

$C_3: (x,y) = (cos(t), sin(t))$

$t \in [\pi, 0]$ $\quad$ $2 - \frac{8}{3}$

$C_4: (x,y) = (t,0)$ $\quad$ $t \in [1,2]$ $\quad$ 0

Example 4 (details, C1)

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} y^2 dx + 3xy dy$$

$$C_1: (x, y) = (2cos(t), 2sin(t)) \qquad t \in [0, \pi]$$

$$(dx, dy) = (-2sin(t), 2cos(t))$$

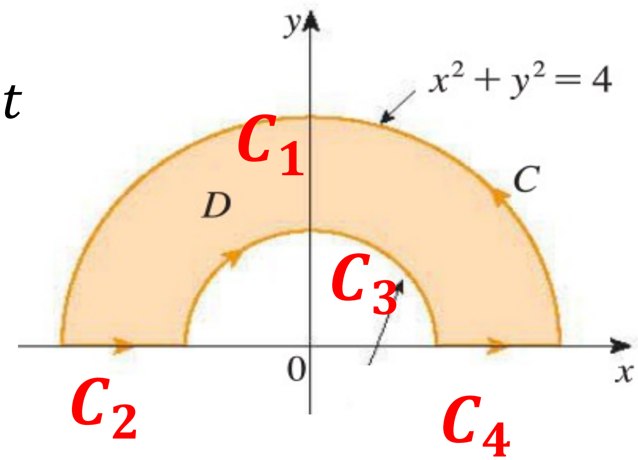
$$= \int_0^{\pi} -8\sin^3(t)dt + 24\cos^2(t)\sin(t)\,dt$$

$$= \int_0^{\pi} -8\,(1 - cos^2(t))\sin(t)dt + 24\cos^2(t)\sin(t)\,dt$$

$$= \int_0^{\pi} -8\sin(t)\,dt + 32\cos^2(t)\sin(t)\,dt$$

$$= [8\cos(t)]_0^{\pi} \quad - \frac{32}{3}[\cos^3(t)]_0^{\pi}$$

$$= -16 + \frac{64}{3}$$



Example 4 (details, C3)

$$\int_{C_3} \vec{F} \cdot d\vec{r} = \int_{C_3} y^2 dx + 3xydy$$

$$C_3: (x, y) = (cos(t), sin(t)) \qquad t \in [\pi, 0]$$

$$(dx, dy) = (-sin(t), cos(t))$$

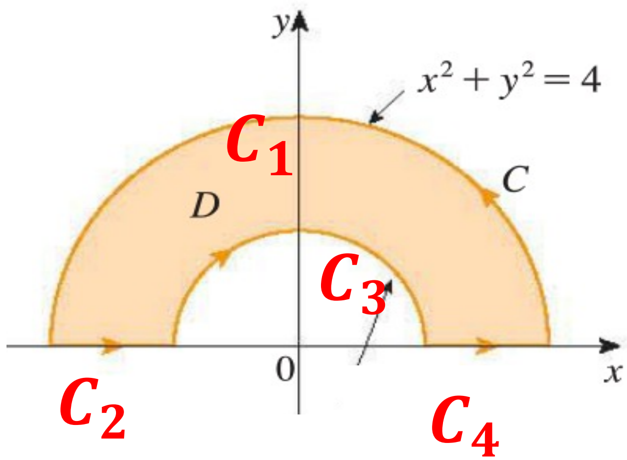
$$= \int_{\pi}^{0} - \sin^3(t)dt + 3\cos^2(t)\sin(t)\,dt$$

$$= \int_{\pi}^{0} - (1 - \cos^2(t))\sin(t)dt + 3\cos^2(t)\sin(t)\,dt$$

$$= \int_{\pi}^{0} - \sin(t)\,dt + 4\cos^2(t)\sin(t)\,dt$$

$$= [\cos(t)]_{\pi}^{0} \quad - \frac{4}{3}[\cos^3(t)]_{\pi}^{0}$$

$$= 2 - \frac{8}{3}$$



Ex5: Whirlpool Field: Double Integral

EXAMPLE 5 If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

$$\iint \nabla \times \vec{F} \cdot \vec{k}\, dxdy = \oint \vec{F} \cdot \vec{T}\, ds = \oint P dx + Q dy$$

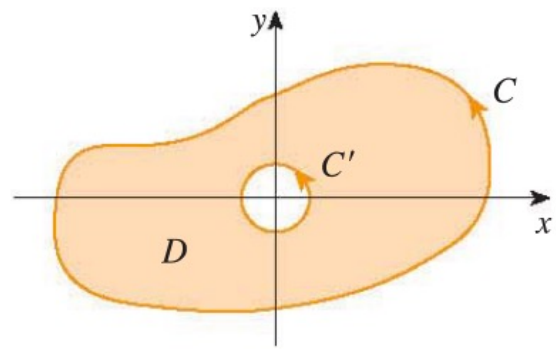
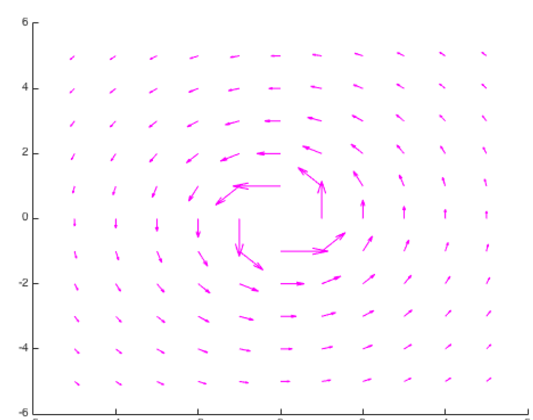


FIGURE 11

$$LHS = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

$$= \iint \left(\frac{y^2 - x^2}{(x^2+y^2)^2} - \frac{y^2 - x^2}{(x^2+y^2)^2}\right) dxdy$$



$$LHS = 0\ ???$$

Solve

Ex5: Whirlpool Field: Line Integral (.continued)

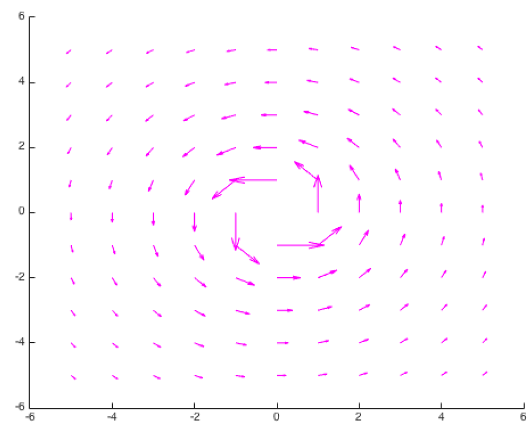
$$\iint \nabla\times\vec{F}\cdot\vec{k}\,dxdy = \oint \vec{F}\cdot\vec{T}\,ds = \oint Pdx + Qdy$$

$$RHS = \oint Pdx + Qdy = \oint \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

$$(x,y) = (a\cos(t), a\sin(t))$$

$$(dx,dy) = (-a\sin(t), a\cos(t))dt$$

3. Whirlpool Field



$$= \int_0^{2\pi} \frac{a^2\sin^2(t)}{a^2}dt + \frac{a^2\cos^2(t)}{a^2}dt$$

$$= \int_0^{2\pi} dt = \boxed{2\pi} = RHS$$

Circulation

Solve

Ex5: Whirlpool Field (with a singularity)

V EXAMPLE 5 If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

$$\iint \nabla \times \vec{F} \cdot \vec{k}\, dxdy = \oint \vec{F} \cdot \vec{T}\, ds = \oint P dx + Q dy$$

$LHS = 0\,???$

$RHS = 2\pi$

Recall $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} =$

$$\left(\frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2} \right)$$

analyze
(discuss)
(interpret)

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \ \textit{except at } (x, y) = (0,0)$$

Failure of existing rules is the prelude to a search for new ones. (Gleick, 1987)

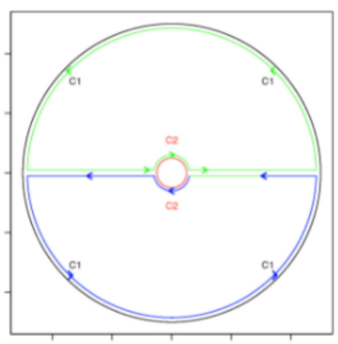
7. **Question Details** m252-supplemental-ex3-final-new-r [4655383] -

Let C be a circle, $x^2+y^2=a^2$, C' be a circle, $x^2+y^2=\varepsilon^2$,
D be the region $0 \le x^2+y^2 \le a^2$, D_s be the region $0 < \varepsilon^2 \le x^2+y^2 \le a^2$, and $\vec{F} = (P, Q)$.
Let $\vec{F}$ be the vector in Eq. A3, *i.e.*, $(P, Q) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$.

(a) Find $\oint_C \vec{F} \cdot \vec{T}\, ds$; ans =

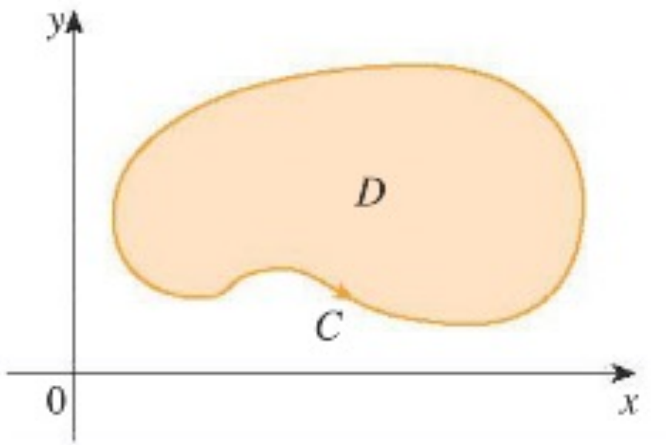
(b) Find $\int\int_{D_s} \nabla \times \vec{F} \cdot \vec{k}\, dA$; ans =

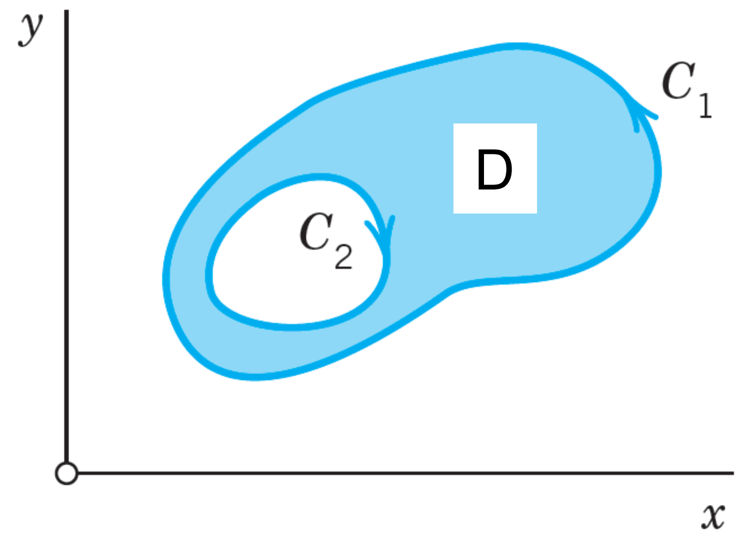
(c) Compare the results in (a) and (b).



Two Cases with Positive Orientation

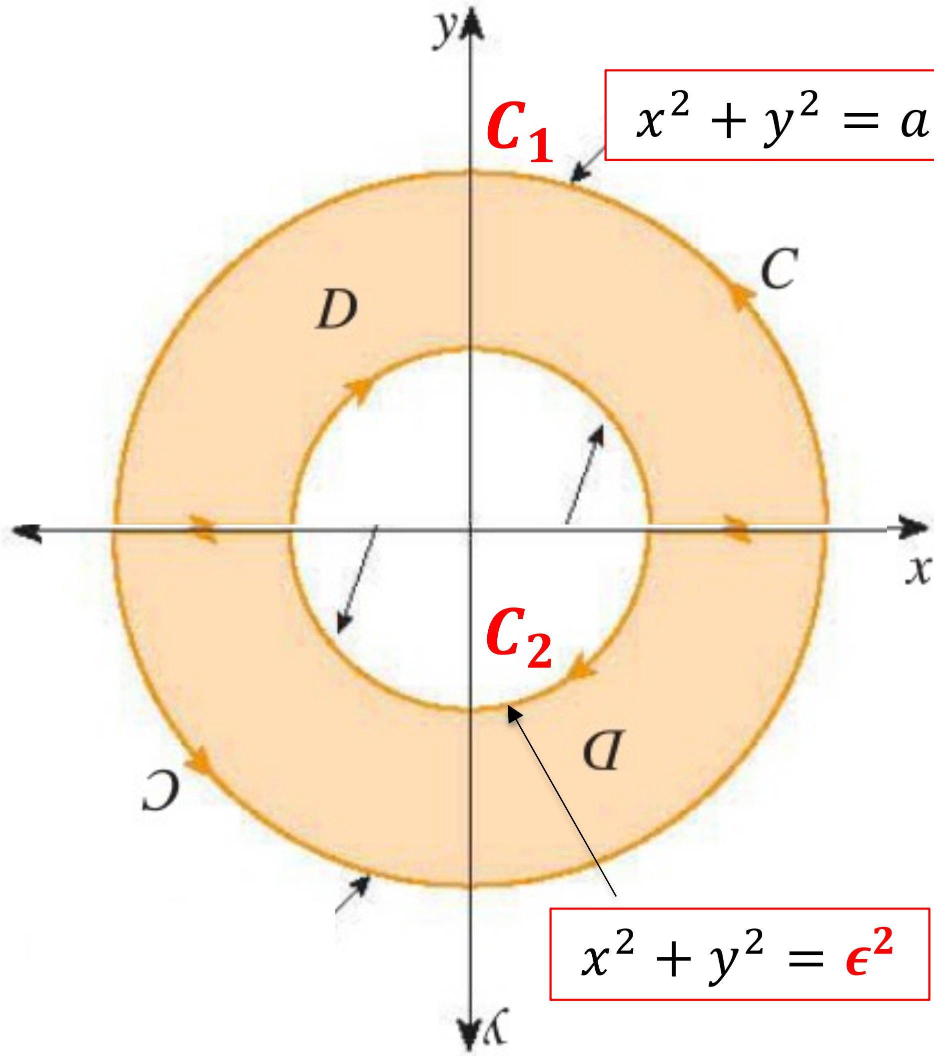
- A simply connected domain (left): Along the entire boundary C of D in such a sense that D is on the left as we advance in the direction of integration.



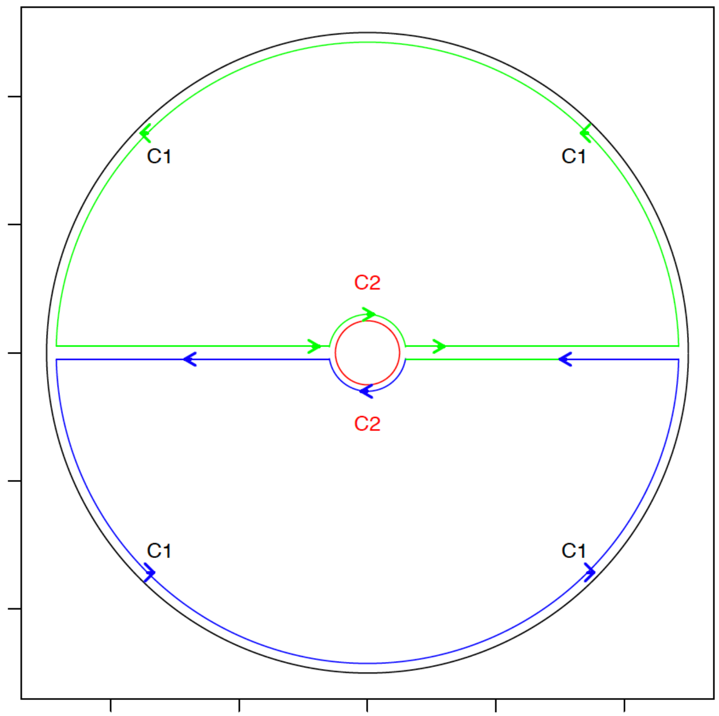


- A region that is not simply connected (right): Region D whose boundary <u>C consists of two parts</u>: C_1 is traversed counterclockwise, while C_2 is traversed clockwise in such a way that D is on the left for both curves.

Annular Ring (for removing a singularity)



$tiny \ \epsilon$

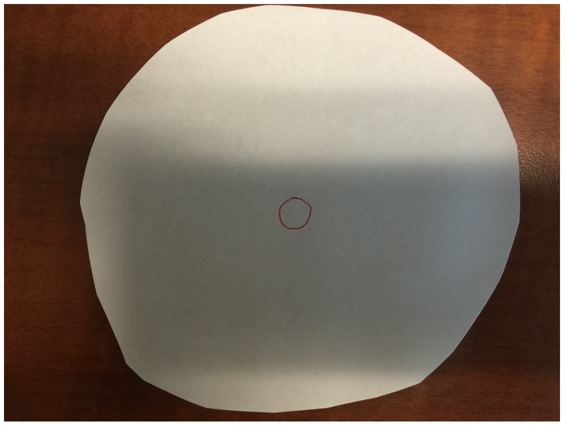
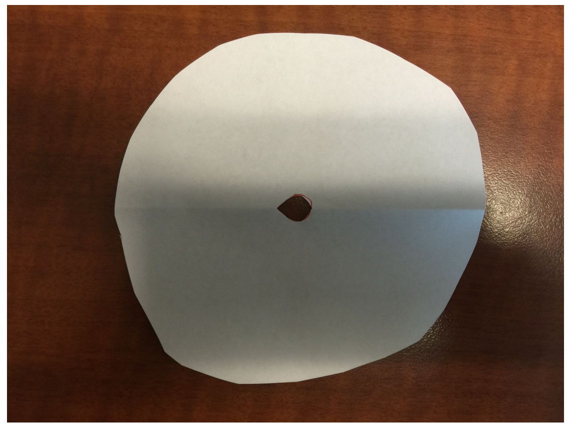
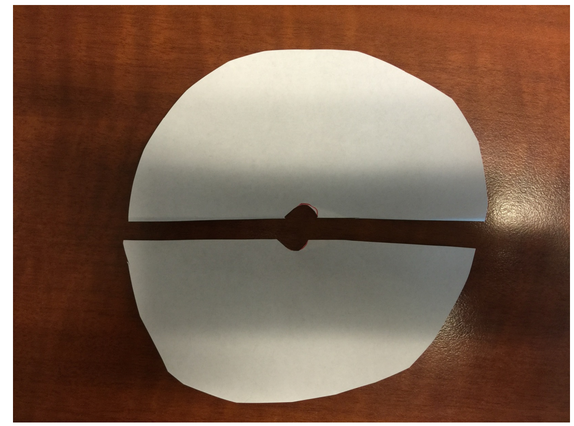


Remove a singularity at the origin

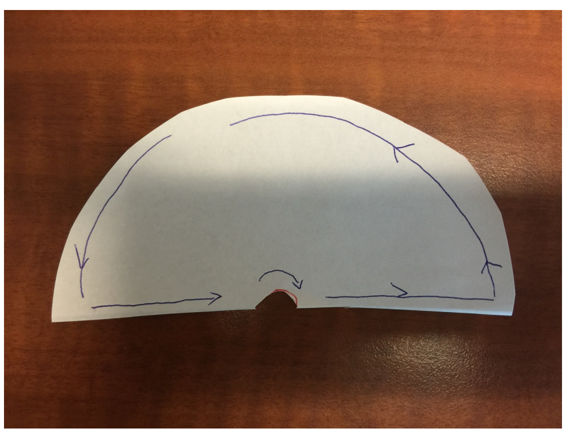
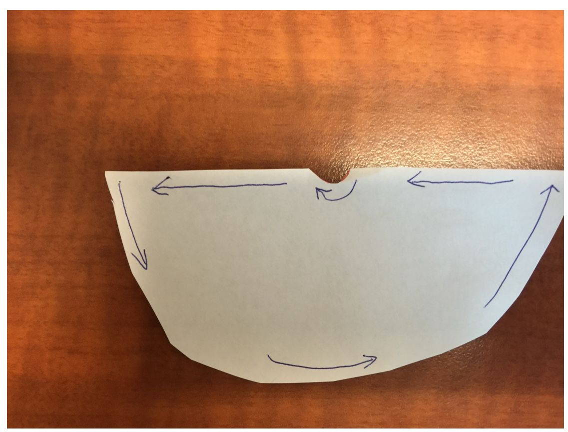
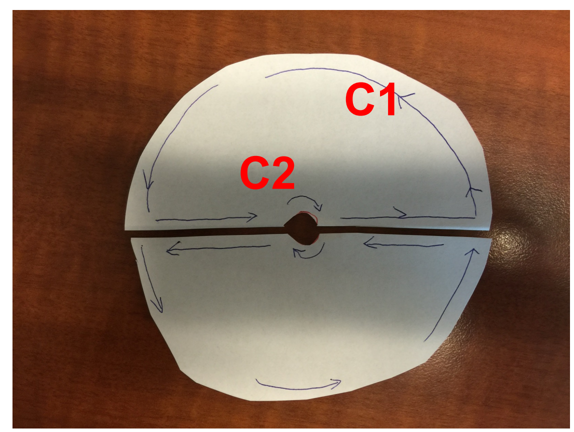
Determine the path/curve for the line integral

When a singularity appears …

$$\oint \vec{F} \cdot d\vec{r} = \int_{C_1+C_2} \vec{F} \cdot d\vec{r}$$

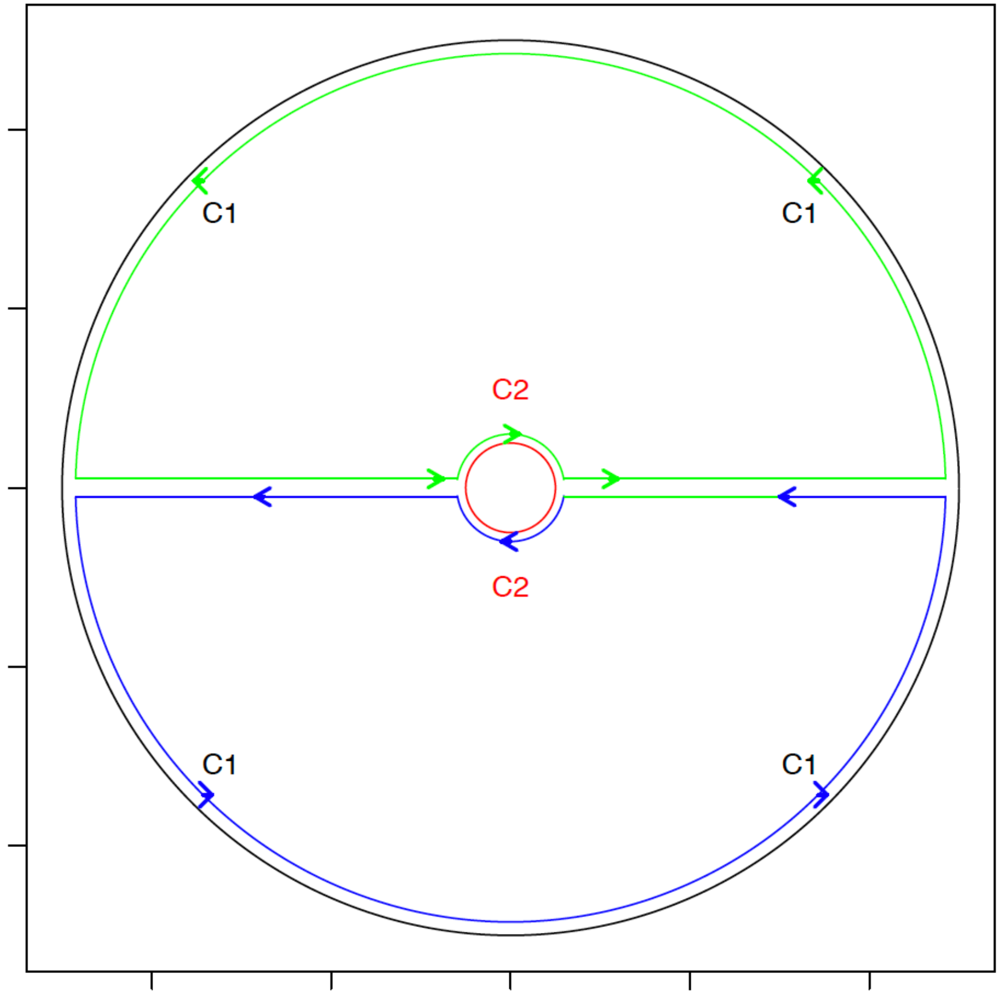
C1=C
C2=-C'

When a singularity appears ...

$$\oint \vec{F} \cdot d\vec{r} = \int_{C_1 + C_2} \vec{F} \cdot d\vec{r}$$

1. Remove the singularity
2. Determine the boundaries
3. Determine the orientation



- A positive orientation leads to $C_1 + C_2$;
- C_1 appears to have a counter − clockwise orientation;
- C_2 appears to have a clockwise orientation;

Ex5: Whirlpool Field (with a singularity)

V **EXAMPLE 5** If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

$$\iint \nabla \times \vec{F} \cdot \vec{k}\, dxdy = \oint \vec{F} \cdot \vec{T}\, ds = \oint P dx + Q dy$$

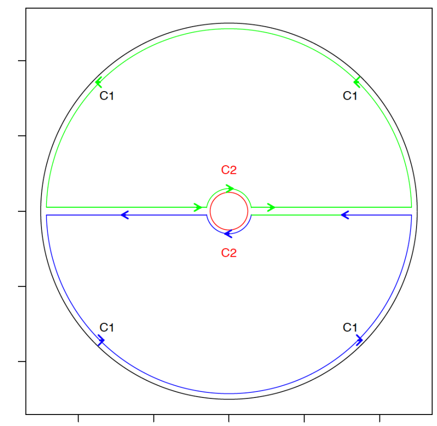
$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \ \ except\ at\ (x, y) = (0,0)$$

Remove the singularity at the origin

$$\mathrm{D_s}:\ \varepsilon < x^2 + y^2 \le a^2$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \ \ in\ D_s$$

$$\iint_{D_s} \nabla \times \vec{F} \cdot \vec{k}\, dxdy = 0$$

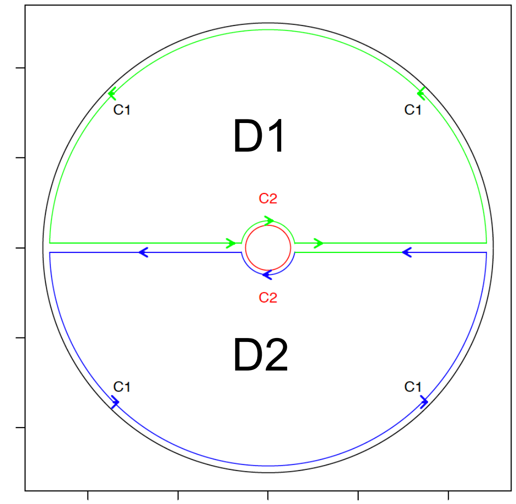


However, the region has “two” boundary curves now.

16.4 Extended Versions of Green's Theorem

The boundary of D1 is illustrated by a green curve, denoted as g*.

The boundary of D2 is illustrated by a blue curve, denoted as b*.



$$\iint_{D_1} \nabla\times\vec{F}\cdot\vec{k}\,dxdy = \int_{g*} \vec{F}\cdot\vec{T}ds$$

$$\iint_{D_2} \nabla\times\vec{F}\cdot\vec{k}\,dxdy = \int_{b*} \vec{F}\cdot\vec{T}ds$$

$$\iint_{D_s} \nabla\times\vec{F}\cdot\vec{k}\,dxdy = \int_{g*} \vec{F}\cdot\vec{T}ds + \int_{b*} \vec{F}\cdot\vec{T}ds = \int_{C_1+C_2} \vec{F}\cdot\vec{T}ds$$

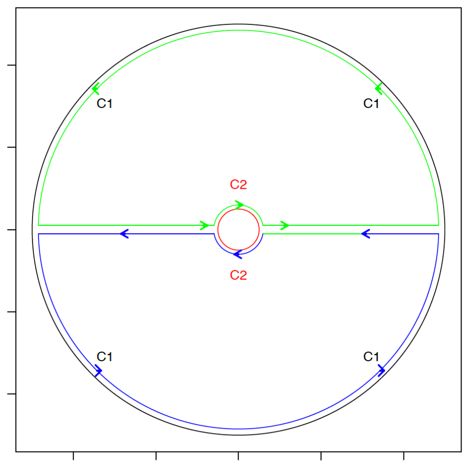
Ex5: Whirlpool Field (with a singularity)

EXAMPLE 5 If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

However, the region has "two" boundary curves now.

$$RHS = \oint \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \boxed{\int_{C_2} \vec{F} \cdot d\vec{r}}$$

$$D_s\text{: } \varepsilon < x^2 + y^2 \le a^2$$



Recall $LHS = \iint_{D_s} \nabla \times \vec{F} \cdot \vec{k}\, dxdy = 0$

Thus, $\int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = 0$ $\quad$ *Let* $C'_1 = -C_2$

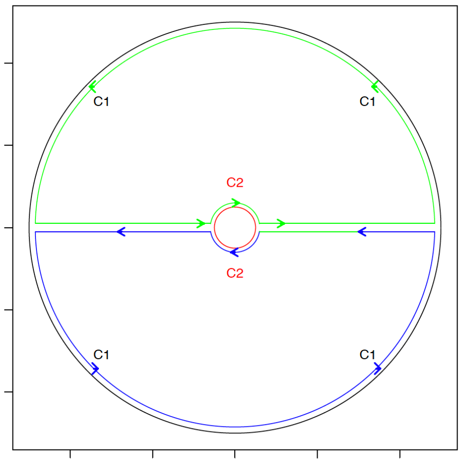
$$\boxed{\int_{C_1} \vec{F} \cdot d\vec{r}} = -\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{-C_2} \vec{F} \cdot d\vec{r} = \boxed{\int_{C'_1} \vec{F} \cdot d\vec{r} = 2\pi}$$

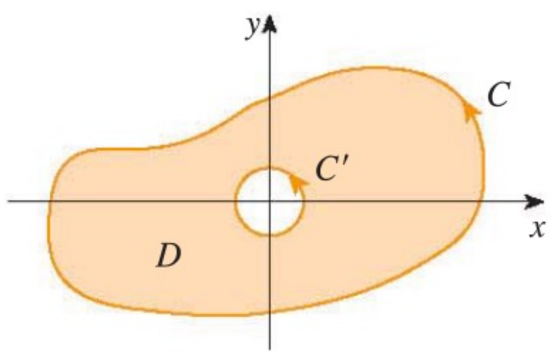
Ex5: Whirlpool Field (with a singularity)

V **EXAMPLE 5** If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

$\mathrm{D_S}$: $\varepsilon < x^2 + y^2 \le a^2$

Consider the following domain





Since $\nabla\times\vec{F} = 0$ and $C_2 = -C'$, we have

$$\iint_{D_S} \nabla\times\vec{F}\cdot\vec{k}\,dxdy = 0$$

$$\iint_{D} \nabla\times\vec{F}\cdot\vec{k}\,dxdy = 0 \quad \int_C \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = 0$$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C'_1} \vec{F}\cdot d\vec{r}$$

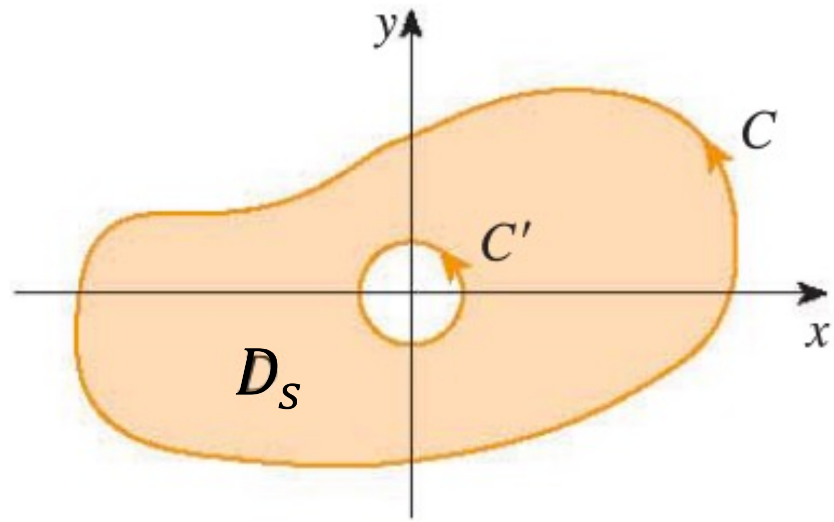
$$\int_{C} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} = 2\pi$$

A Summary for Example 5 (with a singularity)

EXAMPLE 5 If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

$$\iint_{D_S} \nabla\times\vec{F}\cdot\vec{k}\,dxdy = \oint \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\vec{T}ds + \int_{-C'} \vec{F}\cdot\vec{T}ds$$

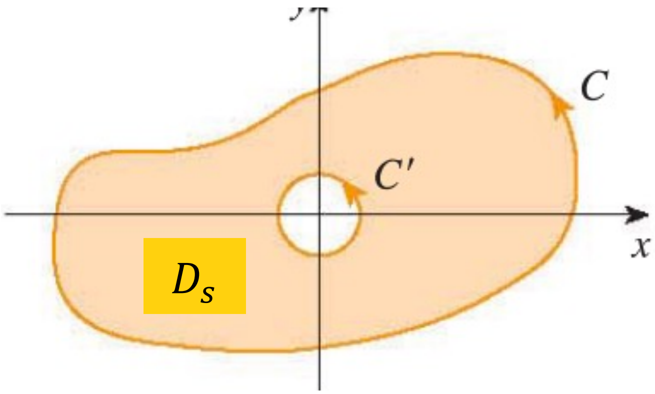
$$\iint_{D_S} \nabla\times\vec{F}\cdot\vec{k}\,dxdy = 0$$



$$\int_C \vec{F}\cdot d\vec{r} = \int_{C'} \vec{F}\cdot d\vec{r} = 2\pi$$

Both C and C′ appear to have counter − clockwise orientation.

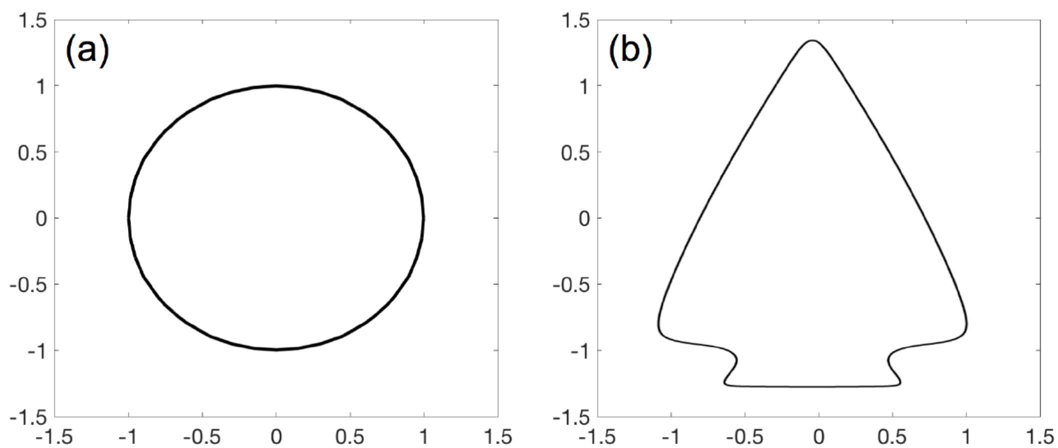
A3: A Summary (Whirlpool Field)



$$\nabla \times \vec{F} \cdot \vec{k} = 0$$

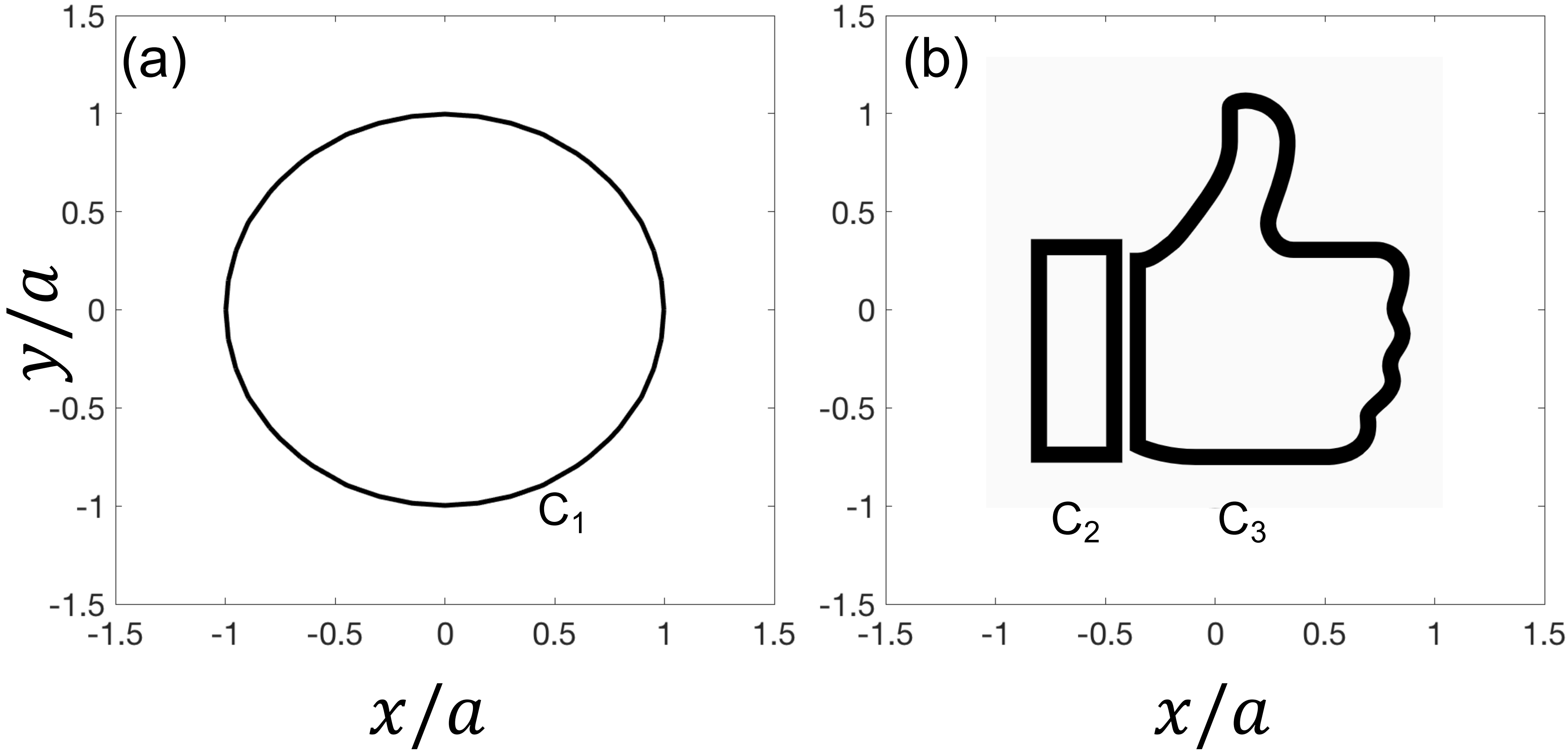
$$\int_C (Pdx + Qdy) = \int_{C'} (Pdx + Qdy)$$

Both C and C' are counterclockwise.



Happy Holidays!

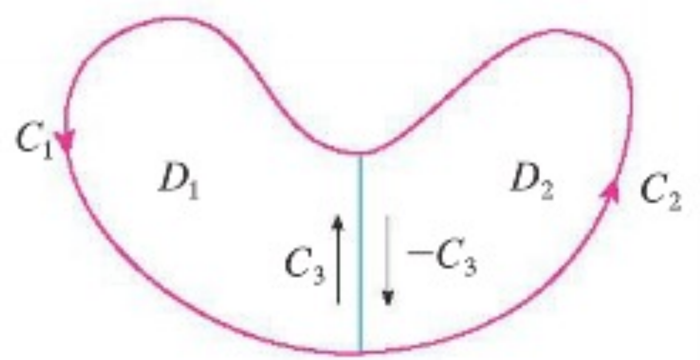
EXAMPLE 5 If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin


(a)
C1
(b)
C2
C3
y/a
x/a
x/a
1.5
1
0.5
0
-0.5
-1
-1.5

16.4 Extended Versions of Green's Theorem

Extended Versions of Green's Theorem

D: a finite union of simple regions.



$$\int_{C_1 \cup C_3} P\,dx + Q\,dy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

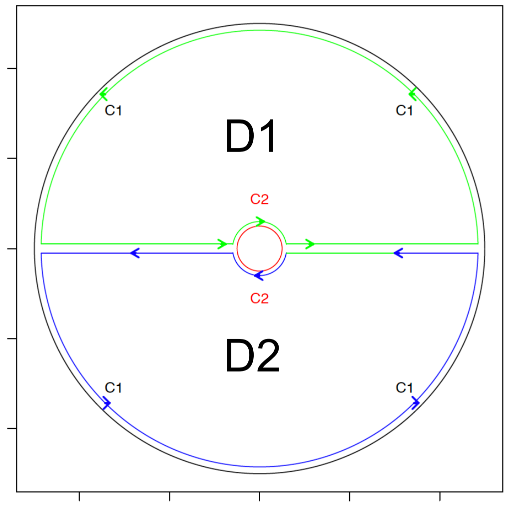
$$\int_{C_2 \cup (-C_3)} P\,dx + Q\,dy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\int_{C_1 \cup C_2} P\,dx + Q\,dy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$D = D_1 \cup D_2$, and its boundary is $C = C_1 \cup C_2$.

The boundary of D_1 is $C_1 \cup C_3$.

The boundary of D_2 is $C_2 \cup (-C_3)$.



The boundary of D1 is illustrated by a green curve.

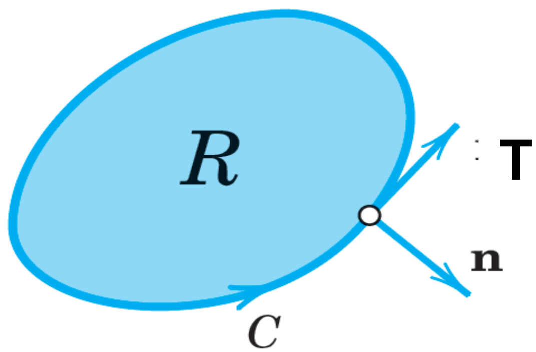
The boundary of D2 is illustrated by a blue curve.

Sumary: Green's Theorem

$$\begin{vmatrix} P & Q \\ dx & dy \end{vmatrix} = Pdx + Qdy$$

$$\oint Pdx + Qdy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

$$\begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ P(x,y) & Q(x,y) \end{vmatrix} = Q_x - P_y$$



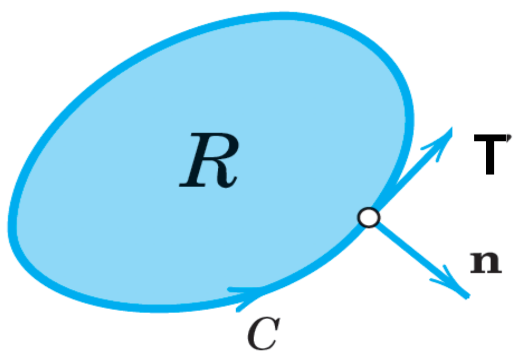
$$\begin{vmatrix} P & Q \\ dx & dy \end{vmatrix} = Pdy - Qdx$$

$$\oint Pdy - Qdx = \iint \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dxdy$$

$$\begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ P(x,y) & Q(x,y) \end{vmatrix} = P_x + Q_y$$

Green's Theorem

circulation (or work) = double integral of "curl"



$$\begin{vmatrix} P & Q \\ dx & dy \end{vmatrix} = P dy - Q dx$$

$$\oint P dy - Q dx = \iint \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx dy$$

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P(x,y) & Q(x,y) \end{vmatrix} = P_x + Q_y$$

flux = double integral of "divergence"

Curl & Divergence (2D Version)

Divergence:

“a Dot product of ∇ and F”

$$
\begin{array}{ccc}
\dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\
\updownarrow & \updownarrow & \\
P(x,y) & Q(x,y) & 0
\end{array}
$$

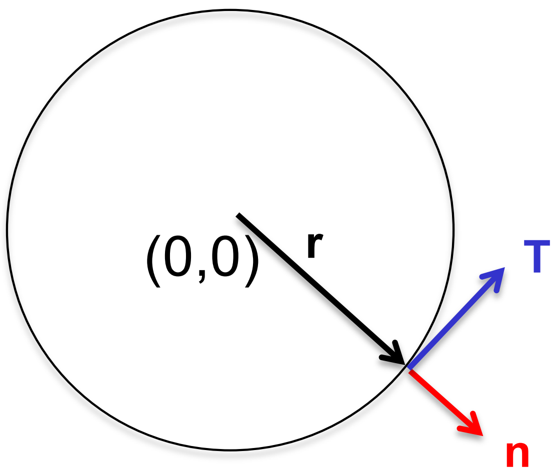
$$\nabla \cdot F = P_x + Q_y$$

5 3 1

convergence

O

Tangential and Normal Vectors



The unit tangent vector is

$$\vec{T} = \left(\frac{dx}{\sqrt{dx^2 + dy^2}}, \frac{dy}{\sqrt{dx^2 + dy^2}}\right)$$

$$\boxed{\vec{T}ds = (dx, dy)}$$ along a (closed) boundary

The unit normal vector is

$$\vec{n} = \vec{T} \times k = \left(\frac{dy}{\sqrt{dx^2 + dy^2}}, \frac{-dx}{\sqrt{dx^2 + dy^2}}, 0\right)$$

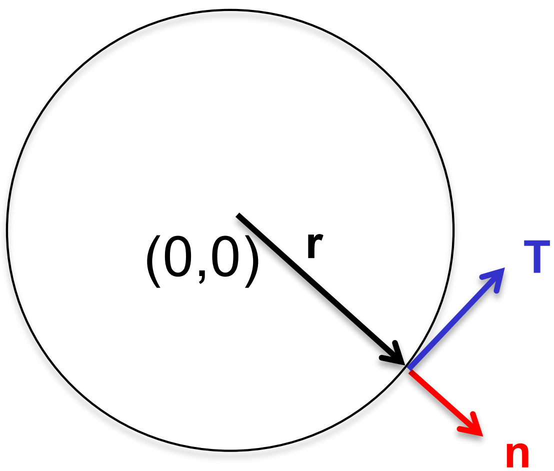
$$\boxed{\vec{n}\, ds = (dy, -dx)}$$ crossing a curve or surface

A parameterization:
r(t) = (x, y)

The derivative of the vector **r** is
dr(t) = (dx, dy)

$$\vec{n} = \vec{T} \times \vec{k} = \frac{1}{ds}\begin{vmatrix} i & j & k \\ dx & dy & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{ds}(dy, -dx, 0)$$

Tangential and Normal Vectors



The unit tangent vector is

$\vec{T} ds = (dx, dy)$ along a (closed) boundary

$+ \quad +$

$$\oint \vec{F} \cdot \vec{T}\, ds = \boxed{\oint P dx + Q dy}$$ circulation

The unit normal vector is

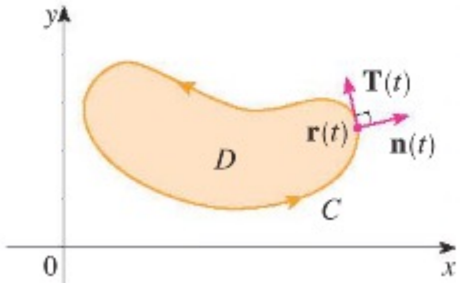
$\vec{n}\, ds = (dy, -dx)$ crossing a curve or surface

$+ \quad -$

$$\oint \vec{F} \cdot \vec{n}\, ds = \boxed{\oint P dy - Q dx}$$ flux

double check: $\vec{T} \cdot \vec{n} = 0$

Circulation (tangential) and Flux (normal)



DEFINITIONS If $\mathbf{r}(t)$ parametrizes a smooth curve C in the domain of a continuous velocity field $\mathbf{F}$, the **flow** along the curve from $A = \mathbf{r}(a)$ to $B = \mathbf{r}(b)$ is

tangential

$$\text{Flow} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds. = \int P dx + Q dy \qquad (5)$$

The integral in this case is called a **flow integral**. If the curve starts and ends at the same point, so that $A = B$, the flow is called the **circulation** around the curve.

DEFINITION If C is a smooth simple closed curve in the domain of a continuous vector field $\vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j}$ in the plane, and if $\mathbf{n}$ is the outward-pointing unit normal vector on C, the **flux** of $\mathbf{F}$ across C is

normal

$$\text{Flux of } \mathbf{F} \text{ across } C = \int_C \mathbf{F} \cdot \mathbf{n}\, ds. = \int P dy - Q dx \qquad (6)$$